MET 3502/5561 Synoptic Meteorology
Lecture 11: Curl, LaPlacian, Total Derivative, and Coordinate Systems

1. Vorticity/curl:

The cross product of ∇ with vector $\vec{A}$ is called the curl of $\vec{A}$:

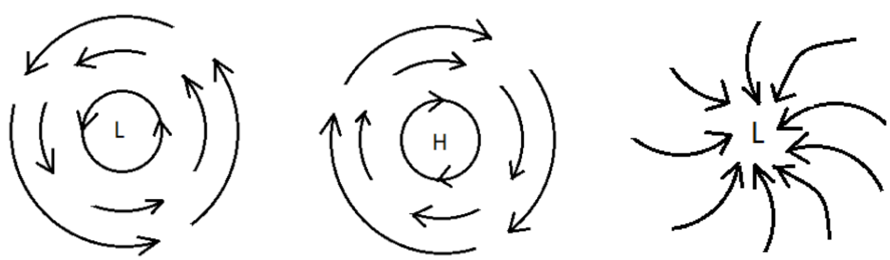
$$\nabla \times \vec{A} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}\right)$$

$$= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

3×3 determinant

The curl of wind vector $\vec{v}$ is called **vorticity**, which is a measure of the rotation of a fluid.

1) Vorticity represents rotation & distortion of an element of air when the wind increases cross stream.
2) A flow with zero curl is said to be irrotational. Positive vorticity is cyclonic (anti-clockwise in Northern Hemisphere), negative vorticity is anticyclonic.



Cyclonic Anticyclonic Convergent & Cyclonic

3) Convergence (divergence) tends to increase (decrease) the curl (vorticity).

2. **Laplacian Operator:**

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Which is the gradient of gradient, or the rate change of gradient (second derivative)

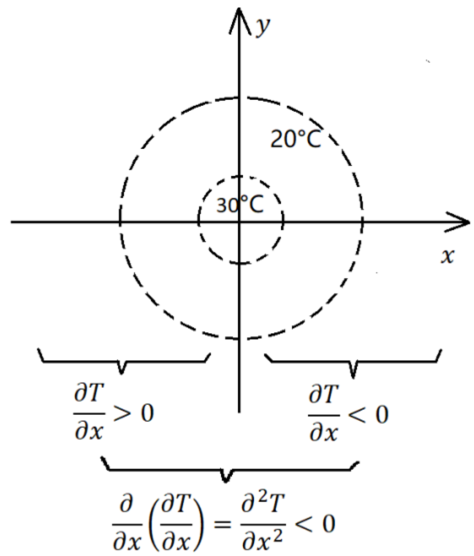
Example: $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

Physical Interpretation:

When there is A minimum in $T \Longrightarrow \nabla^2 T > 0$

When there is A maximum in $T \Longrightarrow \nabla^2 T < 0$

Maximum in T means there is minimum in $\nabla^2 T$



3. **Expansion of a total derivative** (Martin's book, chapter 1.2.4):

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$$

$\frac{D}{Dt}$: Total Derivative; Lagrangian rate of change; Derivative following a parcel; (Rate of change following the flow)

at time 1
at Tampa

at time 2
at Miami

If everything is conserved, $\frac{D}{Dt} = 0$

$\frac{\partial}{\partial t}$: Partial derivative; Euler rate of change; Local rate of change; Derivative at a point; If steady state, then $\frac{\partial}{\partial t} = 0$

$$\frac{\partial T}{\partial t} = \frac{T_2 - T_1}{\Delta t}$$

$+\vec{v} \cdot \nabla$: Negative Advection (remember advection is $-\vec{v} \cdot \nabla$)

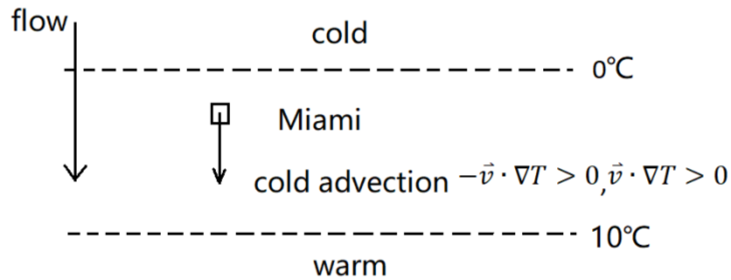
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

For example: $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$

is frequently expressed in pressure coordinates:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x_p} + v\frac{\partial T}{\partial y_p} + \omega\frac{\partial T}{\partial P}$$

Take along pressure surfaces ($\frac{\partial T}{\partial x_p}$, $\frac{\partial T}{\partial y_p}$); $\omega = \frac{DP}{Dt}$



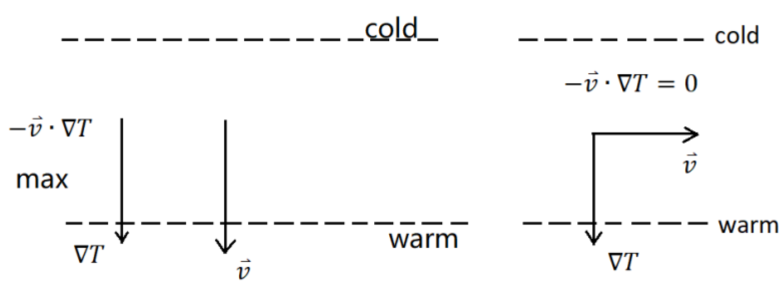
If steady state at Miami, then $\frac{\partial T}{\partial t} = 0$ (No local change, but we do have cold advection ($\vec{v} \cdot \nabla T > 0$) & surface heating ($\frac{DT}{Dt} > 0$) balance each other.)

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T$$

Advection: $-\vec{v} \cdot \nabla$, is based on the gradient of anything and wind direction.

Advection means how much the wind intersects the contour of any parameter; how much the wind wants to transfer.

$$\begin{cases} larger\ wind \rightarrow larger\ advection \\ larger\ gradient\ of\ parameter \rightarrow larger\ advection \end{cases}$$



If ∇T & $\vec{v}$ are parallel, the advection is maximum

If ∇T & $\vec{v}$ are perpendicular, the advection is 0.

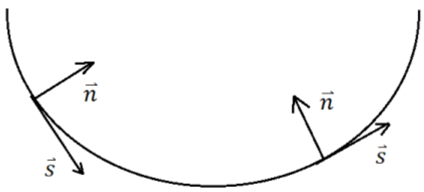
4. **Coordinate Systems:**

1) Cartesian Coordinates: x, y, z (z along constant height surface, x to the east, y to the north)
2) Pressure Coordinates: x, y, P (P along constant pressure surface)

$\frac{\partial}{\partial x_p}, \frac{\partial}{\partial y_p}$ — derivatives are taken along pressure surface, they are almost same as $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, but for math convenience.

3) Isentropic Coordinates: x, y, θ (θ along potential temperature surface). Since θ surfaces are of constant entropy, so (x, y, θ) are referred to as isentropic coordinates. This coordinate is usually for physical interpretation convenience.
4) Natural Coordinates (Martin's book, chapter 4.4) (Wallace & Hobbs's book, chapter 8.1.3): — defined following the flow/ fluid motion (streamlines)

   Advantage: can be useful for diagnosing force balance and the dynamics controlling parcel movement.



streamline
(streamlines are arbitarily spaced lines parallel to $\vec{v}$)

At any point on any horizontal surface, we can define natural coordinates along streamlines:

Unit vectors

- $\vec{n}$:
  - **Direction**: normal to the horizontal flow (perpendicular)
  - **Magnitude**: distance directed normal to the streamline, toward the left of the streamline.
- $\vec{s}$:
  - **Direction**: tangential to the flow
  - **Magnitude**: Arc length directed down stream.

* Unit vectors vary spatially (direction is changing).

So at any point in the flow, $v \equiv |\vec{v}| = \frac{ds}{dt}$, & $\frac{dn}{dt} = 0$.

Momentum Equation: $(\overrightarrow{v} = v\overrightarrow{s})$

$$\frac{D\vec{v}}{Dt} = \frac{d(v\vec{s})}{dt} = \vec{s}\frac{dv}{dt} + v\frac{d\vec{s}}{dt} = \vec{s}\frac{dv}{dt} + \vec{n}\frac{v^2}{R}$$

(R is radius of carvature R > 0 for cyclonic flow)

- $\frac{D\vec{v}}{Dt}$: acceleration of wind vector
- $\frac{d\vec{s}}{dt}$: rate of change of its direction
- $\frac{dv}{dt}$: rate of change of wind speed
- $\frac{v^2}{R}$: centripetal acceleration