MET 3502/5561 Synoptic Meteorology

Lecture 12: Governing Equations

1. What equations govern synoptic-scale atmospheric motions?

1) Momentum Equations — Newton's second law for x & y components of wind
2) Hydrostatic Equation — Newton's first and second laws applied to the z component of wind
3) Gas law specialized for dry air
4) Conservation of heat energy (not cover in here)
5) Conservation of mass (not cover in here)

2. Momentum Equations: Equations of motion (speed & acceleration)

1) Wind — air in motion relative to the earth surface
2) Newton's Laws of motion:
   a) The 1st Law: Every object continues in a state of rest or straight line motion at a fixed speed unless some force acts upon it.
      Example: roll a ball on a table, the ball will stop eventually by friction.
   b) The 2nd Law: The rate of change of velocity (speed and/or direction) of an object is proportional to the force acting on that object divided by the mass of the object:

$$acceleration: a = \frac{d\vec{V}}{dt} = \frac{\sum Forces}{M}$$

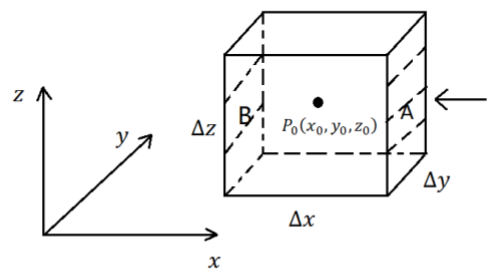
3. Earth-based coordinate system is special (rotating):

1) Newton's 2nd Law is valid only for motions measured in a non-accelerating coordinate system.
2) Earth-based coordinate system (x – longitude, y – latitude, z – height above sea level): Earth is rotating on its axis and revolves around the sun. ⇒ Earth-based coordinates are accelerating.
3) So Newton's 2nd Law can only be applied to the motion of objects on Earth if we correct for the acceleration of our coordinate system.

4. Two categories of forces: (Martin's book, Chapter 2)

1) Fundamental forces: are those forces that would affect objects even in absence of Earth's rotation
   a) Pressure gradient force (PGF)
   b) Gravitational force
   c) Frictional force
2) Apparent forces: are those for correcting the acceleration of our Earth coordinate system.
   a) Centrifugal force ($\xrightarrow{to\ balance}$ Centripetal acceleration)
   b) Coriolis force

5. The pressure gradient force (PGF): derivation of PGF assuming the pressure gradient is known.



Consider the pressure exerted by the atmosphere on sides A & B of the infinitesimal fluid element with the pressure at the center being P_0, pressure on side A being P_A and pressure on side B being P_B:

x – direction first:

$$P_A = P_0 + \frac{\partial P}{\partial x} \cdot \frac{\Delta x}{2}$$

(because pressure gradient $\frac{\partial P}{\partial x} = \frac{P_A - P_0}{\frac{\Delta x}{2}}$)

$$P_B = P_0 - \frac{\partial P}{\partial x} \cdot \frac{\Delta x}{2}$$

(because pressure gradient $\frac{\partial P}{\partial x} = \frac{P_0 - P_B}{\frac{\Delta x}{2}}$)

Since pressure is defined by pressure force per unit area, then the pressure forces on sides A & B are:

$$F_{A_x} = -P_A \cdot area_A = -\left(P_0 + \frac{\partial P}{\partial x} \cdot \frac{\Delta x}{2}\right) \cdot \Delta y \cdot \Delta z$$

$$F_{B_x} = P_B \cdot area_B = -\left(P_0 - \frac{\partial P}{\partial x} \cdot \frac{\Delta x}{2}\right) \cdot \Delta y \cdot \Delta z$$

So, the net x-direction pressure force exerted on the fluid elements is:

$$F_x = F_{A_x} + F_{B_x} = -\left(P_0 + \frac{\partial P}{\partial x} \cdot \frac{\Delta x}{2}\right) \cdot \Delta y \Delta z + \left(P_0 - \frac{\partial P}{\partial x} \cdot \frac{\Delta x}{2}\right) \Delta y \Delta z$$

$$= -2 \cdot \frac{\partial P}{\partial x} \cdot \frac{\Delta x}{2} \cdot \Delta y \cdot \Delta z = -\frac{\partial P}{\partial x} \cdot \Delta x \cdot \Delta y \cdot \Delta z$$

Based on Newton's 2nd Law, the acceleration due to the net force in x direction is:

$$a = \frac{F_x}{M} = -\frac{\partial P}{\partial x} \frac{\Delta x \cdot \Delta y \cdot \Delta z}{M}$$

Volume $V = \Delta x \Delta y \Delta z$

Density $\rho = \frac{M}{V} = \frac{M}{\Delta x \cdot \Delta y \cdot \Delta z}$

$\Longrightarrow a = \frac{F_x}{M} = -\frac{1}{\rho} \frac{\partial P}{\partial x}$

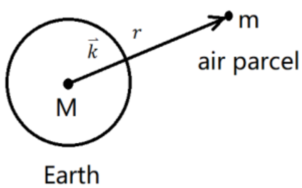
Same derivation for y & z direction, then the total pressure gradient force per unit mass can be expressed as:

$$a = \frac{\vec{F}}{M} = -\frac{1}{\rho}\left(\frac{\partial P}{\partial x}\vec{\imath} + \frac{\partial P}{\partial y}\vec{\jmath} + \frac{\partial P}{\partial z}\vec{k}\right) = -\frac{1}{\rho}\nabla P$$

(Because $\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$)

6. The Gravitational Force (Gravity)

1) Gravity is the force that pulls all objects toward the earth.
2) Gravity is due to the universe attraction of any two elements of mass, in this case, earth & a fluid parcel of air.
3) Gravity is proportional to their masses and inversely proportional to their distance (according to the Law of Universal Gravitation by Newton):



$\overrightarrow{F_g} = -\frac{GMm}{r^2}\vec{k}$

M — earth's mass, m — air parcel's mass

r — distance between the center of their masses, $\vec{k}$ — unit direction of attraction force, and G is the Gravitational constant

According to Newton's 2nd Law:

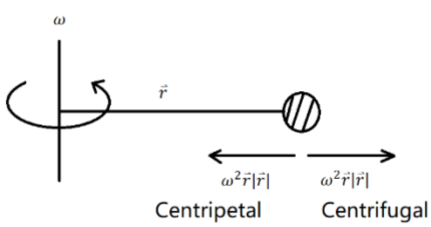
The acceleration of air parcel due to gravity is $g = \frac{F_g}{m} = -\frac{GM}{r} = -9.8\ m/s^2$

7. The frictional force:

1) Important only near surface
2) Always acts to slow the wind
3) 1st order approximation is:
$$a_x = -ku, \qquad a_y = -kv$$
4) Small. → We don't consider it in synoptic meteorology.

8. Centrifugal force (Martin chapter 2.2.2; Holton chapter 1.5.1)

is the acceleration due to Earth's rotation



In order to balance the centripetal acceleration to apply Newton's laws, the centrifugal force is directed outward along the radius of rotation.

Here ω is Earth's rotation speed (angular velocity). Later in the Coriolis force, we'll use Ω for this constant as well.

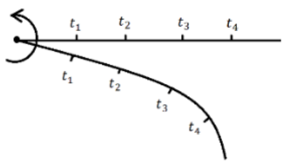
$$CEN = \omega^2\vec{r} = \left(\frac{v^2}{r^2}\right)\cdot\vec{r}|\vec{r}| = \frac{v^2}{r}\vec{r}$$

Note: 1) Centrifugal force is small

2) Generally combined with gravity in dynamics, so we don't consider it explicitly.

9. Coriolis force (Martin 2.2.2; Holton chapter 1.5.1)

1) Also due to Earth rotation
2) Affects only moving objects
3) Deflect to the right in the Northern Hemisphere
   Deflect to the left in the Southern Hemisphere
4) Coriolis force is always much weaker than gravity, so only the part that acts in the local horizontal is significant.
5) Expression:
   a. Why does Coriolis force can deflect the moving objects?
      Assume an object is moving in a straight-line motion in a not-rotating system. Then if the object is observed from a rotating system with the axis of rotation perpendicular to the plane of motion, the path appears to be carved.



Thus, as viewed in a rotating coordinate system, there is an apparent force that deflects an object from a straight-line path

(Merry-go-round example for explaining Coriolis force: A ball thrown between two friends on a merry-go-round will appear to them to take a curved path. They are spinning with the merry-go-round, while the ball moves freely through the air. The force that causes this curvature of motion in the rotating reference frame is the Coriolis force.)

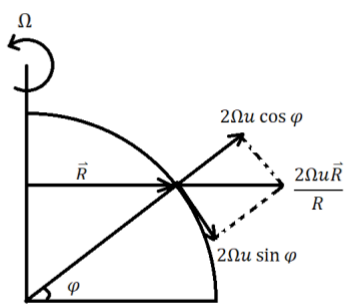
   b. For a small air parcel on the rotating earth system:

Condition A: Parcel at rest.

- The only forces acting on it are the gravity and the apparent centrifugal force ($\Omega^2\vec{R}$) due to the rotation of the earth.

Condition B: Parcel is moving eastward by an impulsive force

- Since now the parcel is rotating faster than the earth, the centrifugal force acting on the parcel will be increased.



$$CEN = \left(\Omega + \frac{u}{R}\right)^2 \vec{R} = \underbrace{\Omega^2\vec{R}}_{\text{Centrifugal force (included in gravity)}} + \underbrace{\frac{2\Omega u\vec{R}}{R}}_{\text{Coriolis force}} + \underbrace{\frac{u^2}{R^2}\vec{R}}_{|u| \ll \Omega R\text{ so, neglected this term}}$$

This coriolis force $\frac{2\Omega u}{R}\vec{R}$ can be divided into components in vertical and horizontal (meridional) directions.

So: in horizontal direction: $\left(\frac{dv}{dt}\right)_{Co} = -2\Omega u\sin\varphi$

in vertical direction: $\left(\frac{dw}{dt}\right)_{Co} = 2\Omega u\cos\varphi$ (negligible relative to g)

Therefore, a parcel moving eastward in the horizontal plane in Northern Hemisphere is deflected southward by the corilos force, whereas a westward-moving object is deflected northward.

Condition C: A parcel moving equatorward (detailed derivation can be seen in Martin & Holton's textbooks):

We can show:

$$\left(\frac{du}{dt}\right)_{Co} = 2\Omega v \sin\varphi - 2\Omega \cos\varphi\, w$$

Where is $-2\Omega \cos\varphi\, w$ small and negligible, so we get:

$$\left(\frac{du}{dt}\right)_{Co} = 2\Omega v \sin\varphi$$

Combine condition B & C, we have:

$$\left(\frac{du}{dt}\right)_{Co} = 2\Omega v \sin\varphi = fv$$

$$\left(\frac{dv}{dt}\right)_{Co} = -2\Omega u \sin\varphi = -fu$$

f is Coriolis parameter.

$f = 2\Omega \sin\varphi$, where Ω is Earth's rotation speed (angular velocity), φ is latitude.

$$2\Omega = 1.454 \times 10^{-4} s^{-1}$$

f is a function of latitude φ (f increases as φ increases)

At 45° latitude $f = 1.0 \times 10^{-4} s^{-1}$

At 20° latitude $f = 0.5 \times 10^{-4} s^{-1}$

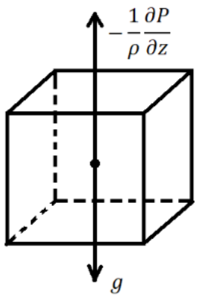
Note: Coriolis force is not important for motions with time scales short compared to the period of earth's rotation (which is 24 hours). So for cumulus clouds, we don't need to consider Coriolis force.

10. Final Equations of motion (acceleration of wind in x, y direction):

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial P}{\partial y}$$

11. Hydrostatic Equation (acceleration of wind in z direction) — Martin chapter 3.1



The atmosphere has mass → $5.265 \times 10^{18}\ kg$.

There is a vertical pressure gradient force given by:

$PGF_{vertical} = -\frac{1}{\rho}\frac{\partial P}{\partial z}$

Which compels the air parcel from high pressure (near surface) to low pressure (above the surface), so it is directed upward. Gravity is balanced by the vertical pressure gradient force, so that the atmosphere doesn't race away into space:

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = g \quad \Longrightarrow \quad \frac{\partial P}{\partial z} = -\rho g$$

This is called hydrostatic approximation. It works well under nearly all conditions except for supercell (tornadoes).

$$\begin{cases} \text{WRF model is a non} - \text{hydrostatic model.} \\ \text{NCEP GFS is a hydrostatic model.} \end{cases}$$

12. Using Gas Law to calculate the density of dry air:

Gas Law: $PV = nR^*T = \frac{M}{m_d}R^*\mathrm{T}$

$\Longrightarrow \qquad P = \frac{M}{V}\cdot\frac{R^*}{m_d}T = \rho R_d T$

$\Longrightarrow \qquad \rho = \frac{P}{R_d T}$

Where R^* is universal gas constant, while all the following parameters are for dry air: M is mass, V is volume, m_d is apparent molecular weight (28.97), T is temperature, P is pressure, ρ is density. R_d is the gas constant for dry air. So for dry air, the density is proportional to P & inversely proportional to T.

13. Hypsometric Equation (Martin chapter 3.1.1)

— provides a relationship between the mean temperature and the thickness of a layer.

Derivation:

From the hydrostatic equation, we have:

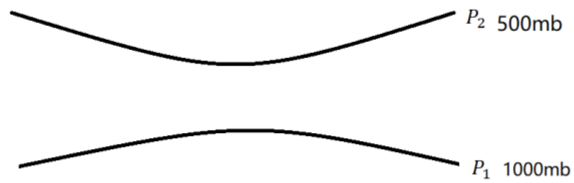
$\frac{\partial P}{\partial z} = -\rho g$, where $\rho = \frac{P}{R_d T}$

$\Rightarrow \frac{\partial P}{\partial z} = -\frac{Pg}{R_d T} \quad \Longrightarrow \frac{dP}{P} = -\frac{g}{R_d T}dz$

Integration between two pressure levels of P_1 and P_2 with height Z_1 and Z_2:

$$\int_{P_1}^{P_2} \frac{dP}{P} = -\frac{g}{R_d} \int_{z_1}^{z_2} \frac{dz}{T}$$

$$\Longrightarrow \ \ln P_2 - \ln P_1 = -\frac{g}{R_d} \int_{z_1}^{z_2} \frac{dz}{T}$$



We know that T is a function of z. Now define a layer mean temperature $\overline{T}$, then:

$$\ln P_2 - \ln P_1 = -\frac{g}{R_d \overline{T}}(z_2 - z_1)$$

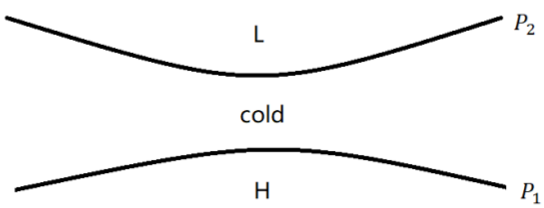
$$\Longrightarrow \ \Delta z = (z_2 - z_1) = -\frac{R_d \overline{T}}{g}(\ln P_2 - \ln P_1)$$

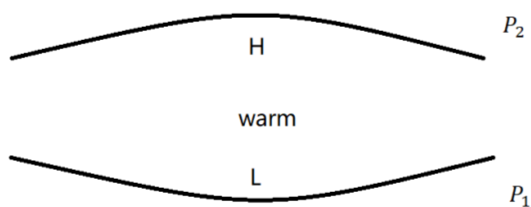
$$\Longrightarrow \text{thickness } \Delta z = \frac{R_d \overline{T}}{g} \ln\left(\frac{P_1}{P_2}\right)$$

Physical Interpretation: $\Delta z \propto \overline{T}$, thickness is proportional to the layer mean temperature.

Large thickness ⟶ warmer mean temperature

Small thickness ⟶ colder mean temperature





14. Scale Height:

Is defined as $H = \frac{R_d \overline{T}}{g}$, which is typically about 6-8 km in troposphere.

Back to hydrostatic equation:

$$\frac{dP}{dz} = -\rho g = -\frac{P}{R_d T} g$$

$$\Longrightarrow \ \frac{1}{P}\frac{dP}{dz} = -\frac{g}{R_d T} \ \Longrightarrow \ \int \frac{dP}{P} = -\frac{g}{R_d} \int \frac{dz}{T}$$

$$\Longrightarrow \ \ln P = -\frac{g}{R_d \overline{T}} \int dz = -\frac{z}{H} + \cos\tan t$$

$$\text{or } \ln \frac{P}{P_0} = -\frac{z}{H}$$

$$\Longrightarrow P = P_0 \exp\left[-\frac{z}{H}\right] = P_0 e^{-\frac{z}{H}}$$

Which states that pressure drops off by a factor of e in passing upward through a layer of depth H.

$\longrightarrow$ so as long as we know the height z, we can get the pressure at that height.