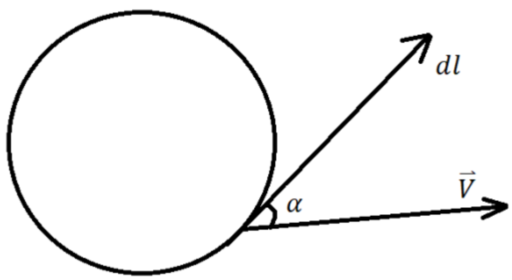
MET 3502/5561 Synoptic Meteorology

Lecture 15: Circulation & Vorticity

Supplementary reading: Holton's textbook, Chapter 4.

Circulation:

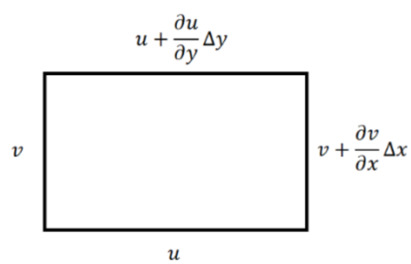
Mathematical Definition: The linear integral about a contour of the component of the velocity vector that is locally tangent to the contour.



$$C \equiv \oint \vec{V} \cdot dl = \oint |V| \cos \alpha \, dl$$

Physical Interpretation: Circulation is a measure of the extent to which a fluid exhibits rotary motion. By convention, it is taken to be positive for counterclockwise integration around the contour.

Component Form Examples: for the circulation of the rectangular fluid contour below:



$$C = \oint \vec{V} \cdot dl = u \cdot \Delta x + \left(v + \frac{\partial v}{\partial x} \cdot \Delta x\right) \cdot \Delta y - \left(u + \frac{\partial u}{\partial y} \cdot \Delta y\right) \cdot \Delta x - v \cdot \Delta y$$

$$= \left[u - u - \frac{\partial u}{\partial y} \cdot \Delta y\right] \cdot \Delta x + \left[v + \frac{\partial v}{\partial x} \cdot \Delta x - v\right] \cdot \Delta y$$

$$= \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] \cdot \Delta x \cdot \Delta y$$

Vorticity:

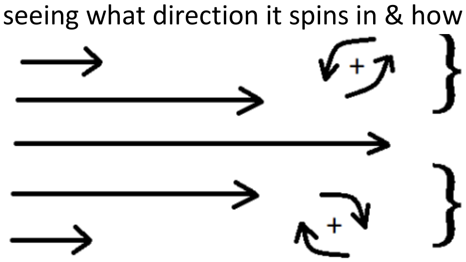
- Defined mathematically as the curl of the velocity vector:

$$\zeta = \nabla \times \vec{V} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{\imath} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\vec{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

- In synoptic meteorology, we typically consider only the vertical component of the vorticity, so that:

$$\zeta = \vec{k} \cdot \nabla \times \vec{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

- This is what synopticians typically refer to as the <u>relative vorticity</u> (add f, then you have absolute vorticity, which is what is typically plotted on synoptic charts).
- <u>What is vorticity</u>?
    - Vorticity is a \$ 0.02 word for <u>spin</u>.
    - Recall $C = \oint \vec{V} \cdot dl = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \cdot \Delta x \cdot \Delta y \Longrightarrow$
      $C = \zeta \cdot A$ or $\zeta = C/A$
    - Vorticity is just a measure of the circulation per unit area.
    - Circulation is a macroscopic measure of rotation, whereas vorticity is a microscopic measure.
- <u>Physical Interpretation</u>:
    - Vorticity is just a measure of spin, so imagine putting a pinwheel in the flow & seeing what direction it spins in & how fast it spins.



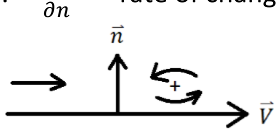
cyclonic vorticity (in NH)

anticyclonic vorticity (in NH)

    - The faster the spin, the greater the magnitude of the vorticity. These are "shear vorticity" (see below).
- <u>Vorticity Components</u>:
    - Another way to think about vorticity is using natural coordinates. In the natural coordinate system, vorticity is given by: $\zeta = \frac{V}{R_s} - \frac{\partial V}{\partial n}$, R_s is the radius of the streamline
    - $\Longrightarrow$ Two components:
        - 1. $\frac{V}{R_s} \longrightarrow$ turning (or curvature) in the wind along a streamline.

This component is called "Curvature Vorticity"

        - 2. $-\frac{\partial V}{\partial n} \longrightarrow$ rate of change of wind speed across a streamline.

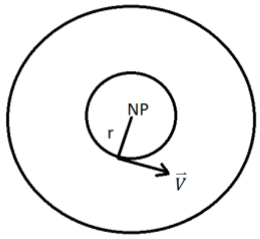


This component is called "Shear Vorticity"

- <u>Relative vs. Absolute Vorticity</u>:
    - So far, we were discussed <u>relative vorticity</u>, which is based on the wind speed relative to a fixed point on earth.

- <u>Absolute Vorticity</u> also considers the effects of the Earth rotation, which is also called the <u>planetary vorticity</u>.

  $\eta = \text{Absolute Vorticity} = \vec{k} \cdot \nabla \times \overrightarrow{V_a} = \vec{k} \cdot \nabla \times \overrightarrow{V_r} + \vec{k} \cdot \nabla \times \overrightarrow{V_e}$, $\overrightarrow{V_r}$ is wind relative to Earth, $\overrightarrow{V_e}$ is the Earth's velocity, $\vec{k} \cdot \nabla \times \overrightarrow{V_e}$ is the vorticity of the earth.
- What is $\nabla \times \overrightarrow{V_e}$ —— the vorticity of the earth?



  - Start by determining the circulation around a latitude belt with latitude value of ϕ:
  - $C = \oint \vec{V} dl$
  - $V = \Omega r$ based on solid body rotation (Ω is the Earth's angular velocity, r is the earth's radius)

  $\Longrightarrow C = \oint \Omega r dl = \Omega r \oint dl$ since Ω & r are constant around line.

  $\oint dl = \text{circumference} = 2\pi r$

  $\Longrightarrow C = \Omega r \cdot 2\pi r = 2\pi r^2 \Omega$
- Recall that vorticity $= C/A \Longrightarrow$

  $$\nabla \times \overrightarrow{V_e} = \frac{C}{A} = \frac{2\pi r^2 \Omega}{\pi r^2} = 2\Omega \longrightarrow \text{Vorticity of solid body rotation \& rotating earth}$$
- But we are interested in the vertical component (i.e. $\vec{k} \cdot \nabla \times \overrightarrow{V_e}$), the component along $\vec{k}$

  $\Longrightarrow \vec{k} \cdot \nabla \times \overrightarrow{V_e} = 2\Omega \sin\phi$ (By geometry! ϕ is the latitude) $= f \longrightarrow$ Coriolis Parameter !

  $\Longrightarrow \eta = \zeta + f = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \qquad \longrightarrow \text{absolute vorticity}$
  - Absolute vorticity is commonly displayed on synoptic charts.
  - Typically expressed in units of $10^{-5} s^{-1}$
  - Rare to see negative absolute vorticity on the synoptic scale