MET 3502/5561 Synoptic Meteorology

Lecture 16: The Vorticity Equation

Supplemental Reading: Holton's book, section 4.4

<u>Motivation</u>: To gain understanding of what cause changes in verticity in order to better understand the development & decay of large scale & mesoscale weather systems. $\left(\frac{\partial \zeta}{\partial t} \text{ \& } \frac{D\zeta}{Dt}\right)$

Derivation:

1. Take $\frac{\partial}{\partial y}$ of the zonal (u) momentum equation & $\frac{\partial}{\partial x}$ of the meridional (v) momentum equation [& ignore friction]:

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x}\right] \quad A$$
$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y}\right] \quad B$$

2. Subtract A from B & substitute $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ to get:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial P}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial P}{\partial x}\right)$$

Here: $\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} = \frac{D\zeta}{Dt}$

$v\frac{\partial f}{\partial y}$: $\frac{Df}{Dt} = \overset{0}{\cancel{\frac{\partial f}{\partial t}}} + u\overset{0}{\cancel{\frac{\partial f}{\partial x}}} + v\frac{\partial f}{\partial y} + w\overset{0}{\cancel{\frac{\partial f}{\partial z}}} \Rightarrow \frac{Df}{Dt} = v\frac{\partial f}{\partial y}$

3. Use $\frac{D\zeta}{Dt}$ & $\frac{Df}{Dt}$ substitutions to obtain the vorticity equation:

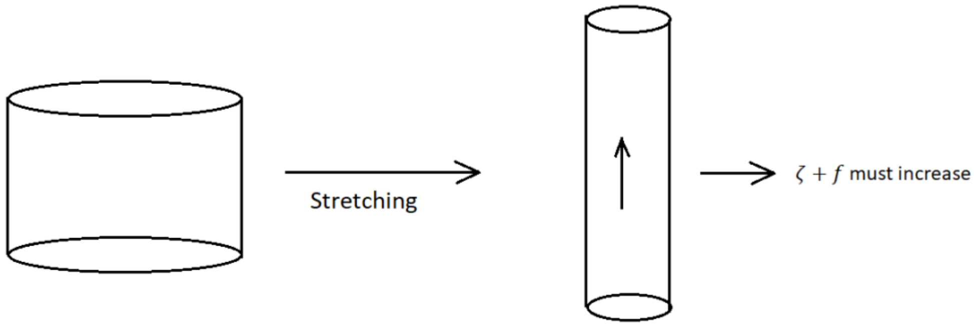
$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial P}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial P}{\partial x}\right)$$

Here:
$\frac{D}{Dt}(\zeta + f)$: rate of change of the vertical component of absolute vorticity
$-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$: Divergence term
$\left(\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial z}\right)$: Tilting term
$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial P}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial P}{\partial x}\right)$: Solenoidal term

<u>Physical Interpretation</u>:

1. <u>Divergence term:</u>
   - Represents the effects of convergence (stretching) & divergence (shrinking) on the absolute vorticity.

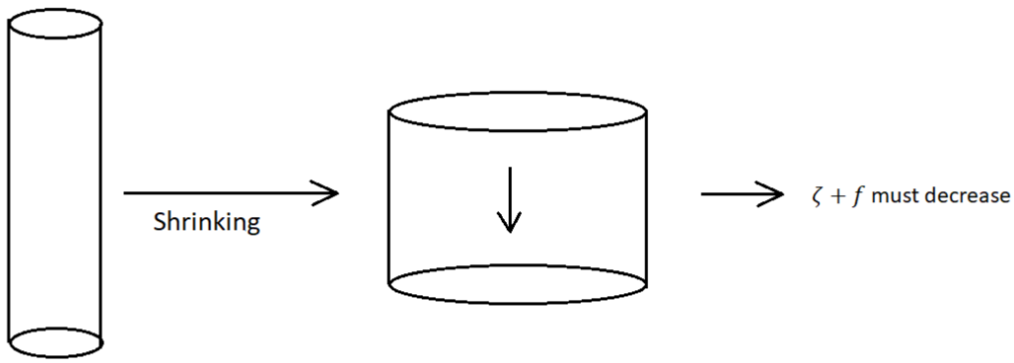
- Convergence (stretching) increases absolute vorticity



$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} < 0$ since: $\frac{D(\zeta+f)}{Dt} \propto -(\zeta+f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \propto -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$

$\Rightarrow \frac{\partial \omega}{\partial z} > 0$ (continuity)

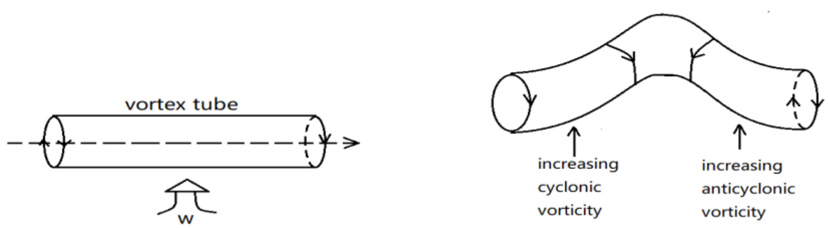
- Divergence (shrinking) decreases absolute vorticity:



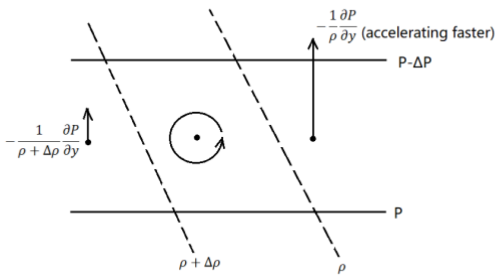
$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} < 0$

$\Rightarrow \frac{\partial \omega}{\partial z} < 0$ (continuity)

2. <u>Tilting term</u>: (twisting or tipping term)
   - Represents the tilting or twisting of horizonal vorticity into the vertical [or vice versa]



   - Important for supercell thunderstorms.
3. <u>Solenoidal term:</u>
   - Only has an effect when the atmosphere is baroclinic (has a horizontal density gradient)
   - This term represents the relationship between pressure gradient force & density. When you push two air parcels with high or low density with same force, low density will be pushed further.
   - For example:



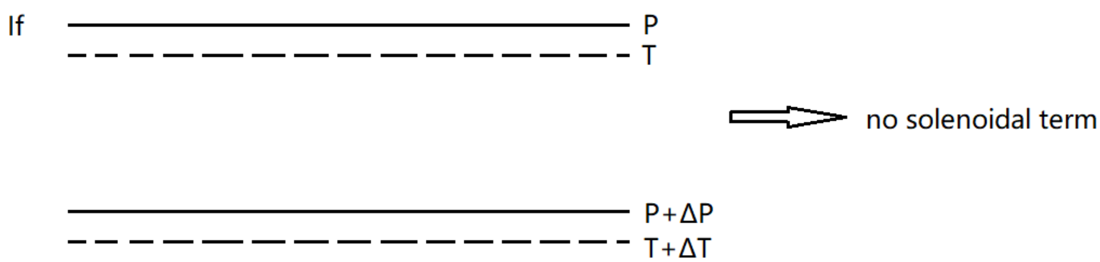
- Although $\frac{\partial P}{\partial y}$ is constant, it accelerates the air on the right more due to the low density.
- As a result, vorticity increases.

- Mathematically:

$$\frac{D(\zeta+f)}{Dt}=\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial P}{\partial y}-\frac{\partial\rho}{\partial y}\frac{\partial P}{\partial x}\right)>0$$ cyclonic production

$\uparrow$ $\downarrow$ $\downarrow$
>0 <0 <0

- You need the different orientation of pressure & temperature (density) gradient to have positive or negative values of the Solenoidal term. Otherwise, if the isobar and isotherm are parallel with each other like the drawing below, then Solenoidal term is equal zero.



- In tornado genesis, the solenoidal term is important.

Scale Analysis of the Vorticity Equation

Motivation: Want to know what terms are important [or dominate] the development & Decay of synoptic systems such as cyclones & anticyclones.

Typical magnitude for synoptic motion:

Field Variable	Symbol	Magnitude (scale)
Horizontal wind speed	U	10 m/s
Vertical velocity	W	1 cm/s $(10^{-2}\ m/s)$
Length	L	1000 km $(10^{6}\ m)$
Depth	H	10 km $(10^{4}\ m)$
Horizonal pressure variability	δP	10 m $(10^{3}\ Pa)$
Density	ρ	1 kg/m^3
Fractional density fluctuation	$\delta\rho/\rho$	10^{-2}
Time	$T=L/U$	$10^5\ s$
Coriolis (at mid-latitude)	f_0	$10^{-4}\ s^{-1}$
β parameter ($\frac{\partial f}{\partial y}$)	β	$10^{-11}m^{-1}s^{-1}$

- Recall vorticity equation:

$$\frac{D}{Dt}(\zeta + f) = (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial P}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial P}{\partial x}\right)$$

1. What is the scale of relative vorticity vs. planetary vorticity?

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \backsim \frac{U}{L} \backsim 10^{-5} s^{-1}$$

$$\text{Thus } \frac{|\zeta|}{|f|} \backsim \frac{|\zeta|}{f_0} \backsim \frac{U/L}{f_0} \backsim \frac{U}{f_0 L} = \text{Rossby Number}$$

For synoptic motions: $R = \frac{U}{f_0 L} = \frac{10^{-5}}{10^{-4}} \backsim 0.1$

<u>Conclusion</u>: For synoptic scale systems, the relative vorticity is an order of magnitude smaller than the planetary vorticity. As a result, ζ can be neglected in the divergence term.

<u>Note</u>: Flows where $R \ll 1$ are called low-Rossby number flows. This means $\zeta \ll f$.

2. What is the magnitude of the divergence term?

$$-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \backsimeq -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\left|-f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right| \backsim f_0 \frac{U}{L} = 10^{-4} \times \frac{10}{10^{-6}} = 10^{-9}$$

However, because $\frac{\partial u}{\partial x}$ & $\frac{\partial v}{\partial y}$ are of opposite signs but nearly equal magnitude, $\left|\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right| \neq \frac{U}{L}(10^{-5})$, but close to 10^{-6}.

$\Longrightarrow \left|-f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right| \backsim 10^{-10}$

3. What is the magnitude of the tilting term?

$$\left|\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial z}\right| \backsim \frac{W}{L}\frac{U}{H} \backsim \frac{10^{-2}}{10^{6}}\frac{10}{10^{4}} \backsim 10^{-11}$$

$\Longrightarrow$ On the synoptic scale, the tilting term is an order of magnitude smaller than the divergence term & can be neglected.

4. What is the magnitude of the solenoidal term?

$$\left|\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial P}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial P}{\partial x}\right)\right| \backsim \frac{1}{\rho}\frac{\delta \rho}{\rho L}\frac{\delta P}{L} = \frac{1}{1}10^{-2}\frac{1}{10^{6}}\frac{10^{3}}{10^{6}} \backsim 10^{-11}$$

$\Longrightarrow$ The solenoidal term can be neglected since it is also an order of magnitude smaller than the divergence term.

5. What are the magnitude of terms in $\frac{D}{Dt}(\zeta + f)$?

$$\frac{D}{Dt}(\zeta + f) = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + \overset{0}{\cancel{\frac{\partial f}{\partial t}}} + u\overset{0}{\cancel{\frac{\partial f}{\partial x}}} + v\frac{\partial f}{\partial y} + w\overset{0}{\cancel{\frac{\partial f}{\partial z}}}$$

$$= \frac{\partial \zeta}{\partial} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + v\frac{\partial f}{\partial y}$$

$\left|\frac{\partial \zeta}{\partial t}\right|, \left|u\frac{\partial \zeta}{\partial x}\right|, \left|v\frac{\partial \zeta}{\partial y}\right| \backsim \frac{U^2}{L^2} \backsim \frac{10^2}{(10^6)^2} \backsim 10^{-10}$

$\left|w\frac{\partial \zeta}{\partial z}\right| \backsim W\frac{U}{LH} \backsim 10^{-2}\frac{10}{10^6 \cdot 10^4} \backsim 10^{-11} \longrightarrow$ Neglect

$\left|v\frac{\partial f}{\partial y}\right| \backsim U \cdot \beta \backsim 10 \cdot 10^{-11} \backsim 10^{-10}$

$\Longrightarrow \frac{D(\zeta+f)}{Dt} \backsim \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + v\beta \backsim \frac{D_h}{Dt}(\zeta + f)$

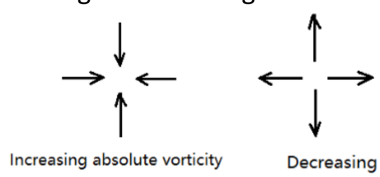
Where $\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$

- Based on 1-5, we get an approximate of the vorticity equation for synoptic scale motions:

$$\frac{D_h}{Dt}(\zeta + f) = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Physical Interpretation:

1. For synoptic-scale flows, the tilting & solenoidal terms can usually be neglected.
2. The vertical advection of vorticity can also be neglected.
3. Changes in the absolute vorticity following a parcel are produced by horizontal convergence & divergence.



4. Not valid on mesoscale weather systems.