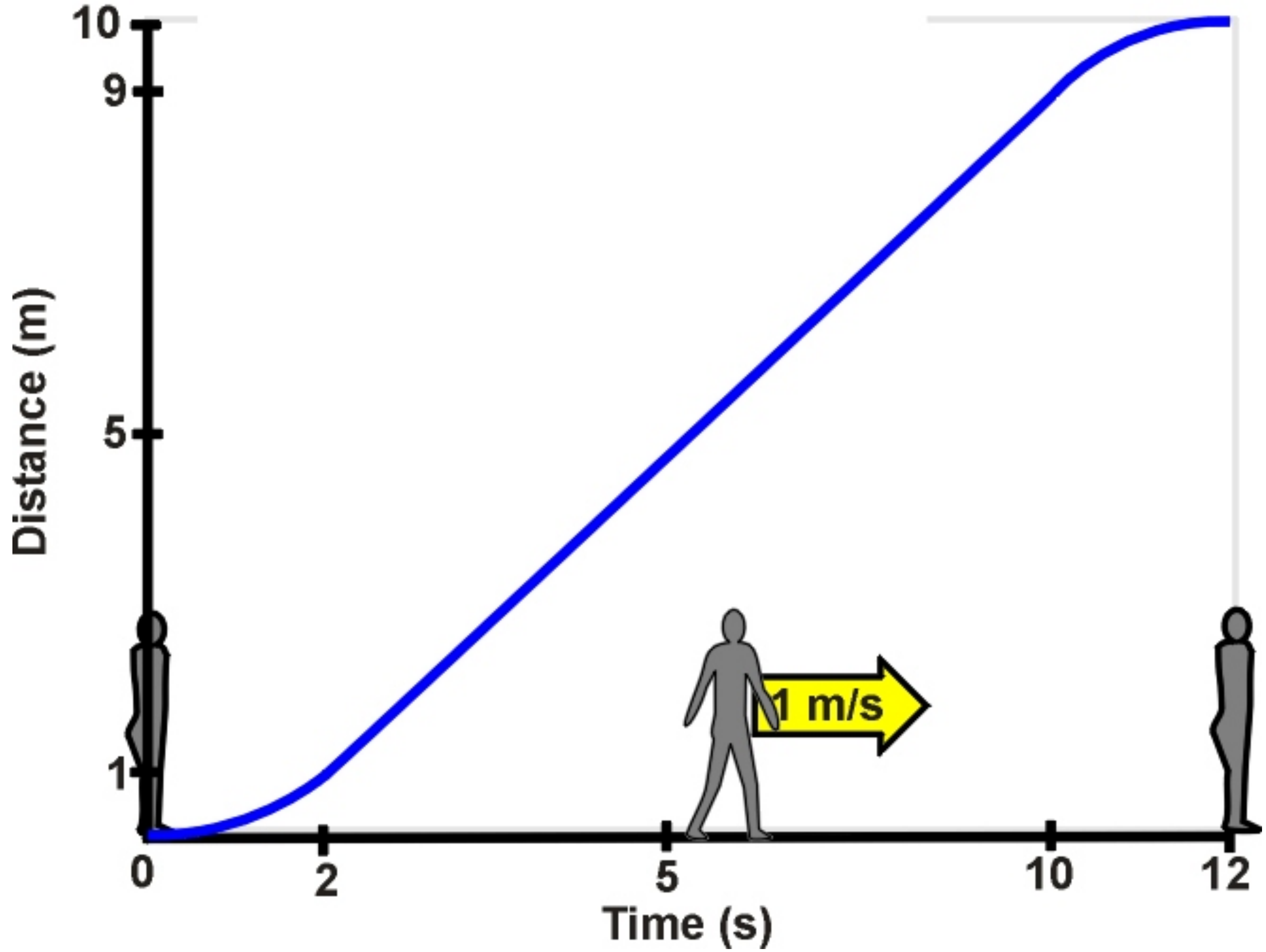
MET 3502/5561
Synoptic Meteorology

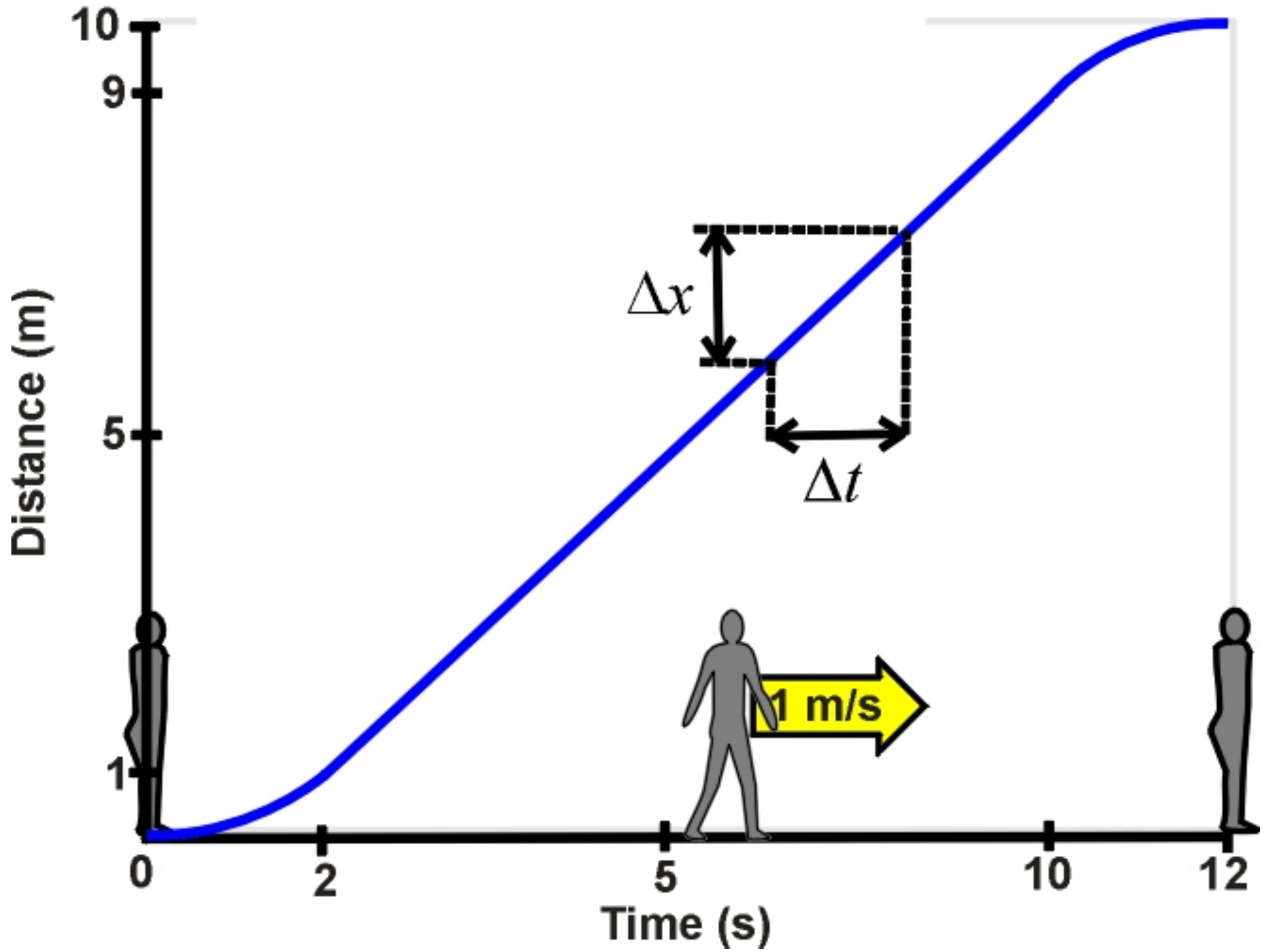
Lecture 9: Math Review:

Derivatives and Integration

Professor Takes a Stroll

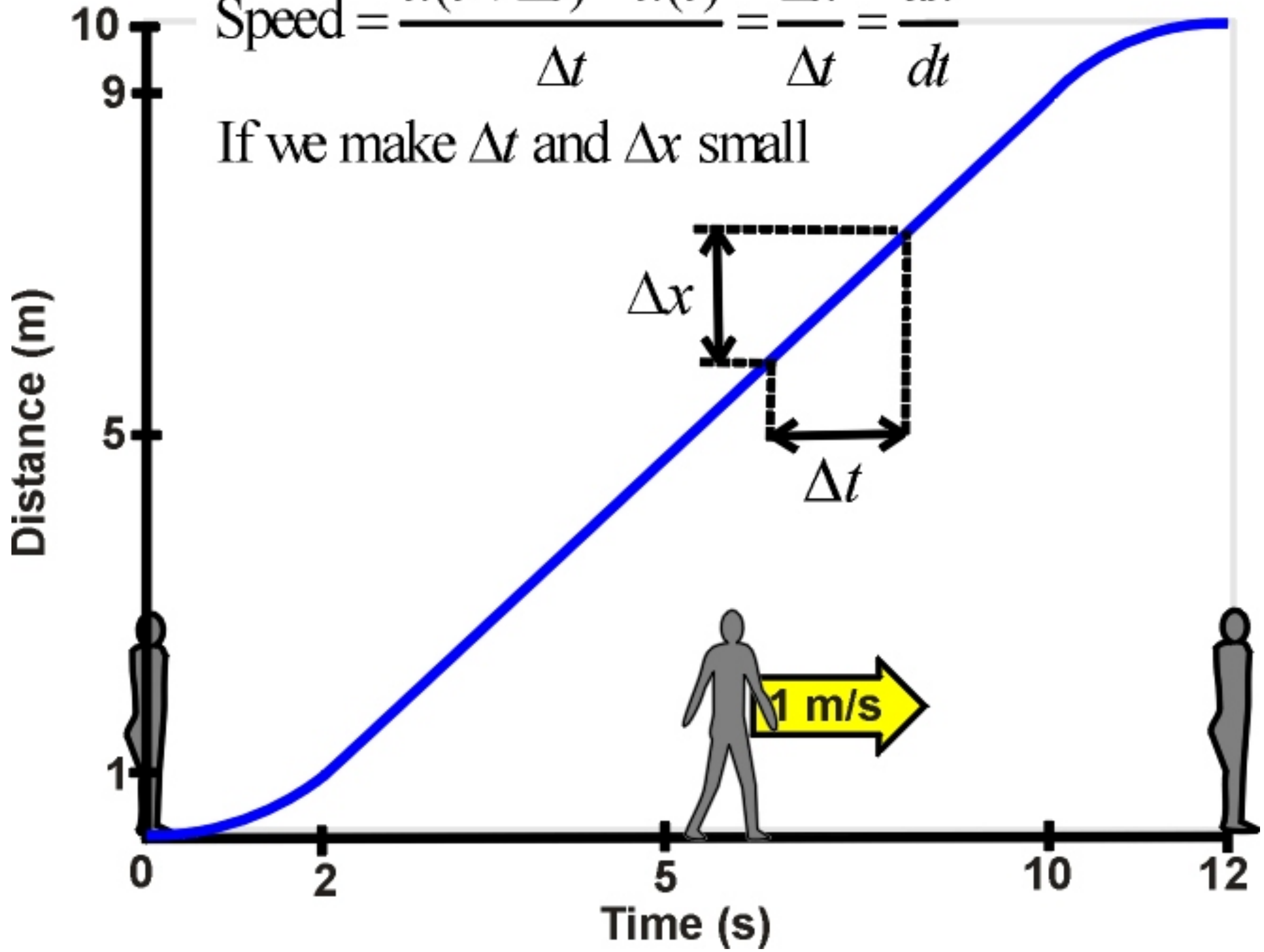


How do you measure his speed?

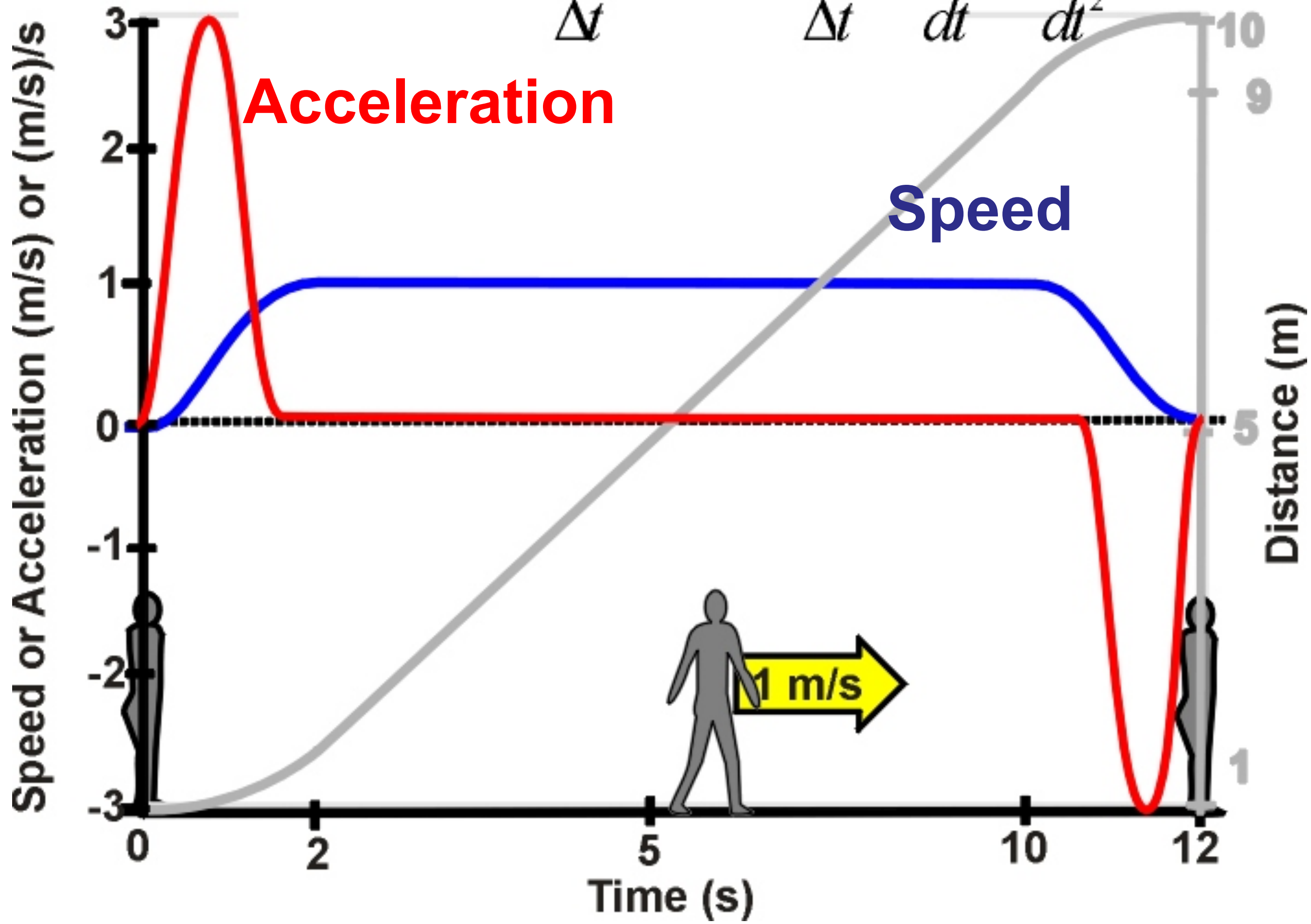


$$\text{Speed} = \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

If we make Δt and Δx small



$$\text{Acceleration} = \frac{u(t+\Delta t) - u(t)}{\Delta t} = \frac{\Delta u}{\Delta t} = \frac{du}{dt} = \frac{d^2x}{dt^2}$$



About Derivatives

- They are a way of calculating the rate of change of a known function.
- Most powerful physical laws—including those that describe the atmosphere---are formulated in terms of derivatives
- There are well-defined ways of going from known derivatives back to the function that they represent
- First, we'll focus on derivatives…

For example, the derivative of x^2

$$\frac{dx^2}{dx} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= 2x + \Delta x \cong 2x$$

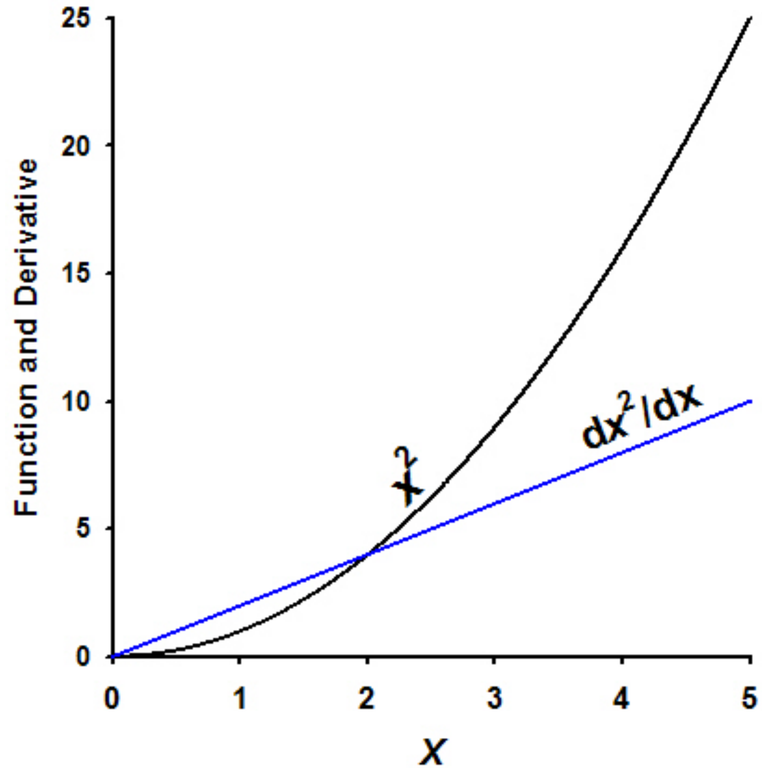
Derivatives of Powers

- Derivatives of x^n work the same way
- $dx^n/dx = nx^{n-1}$
- Works even if n is negative or not a whole number
- For example

  $$dx^{½}/dx = ½\,(1/x^{½})$$

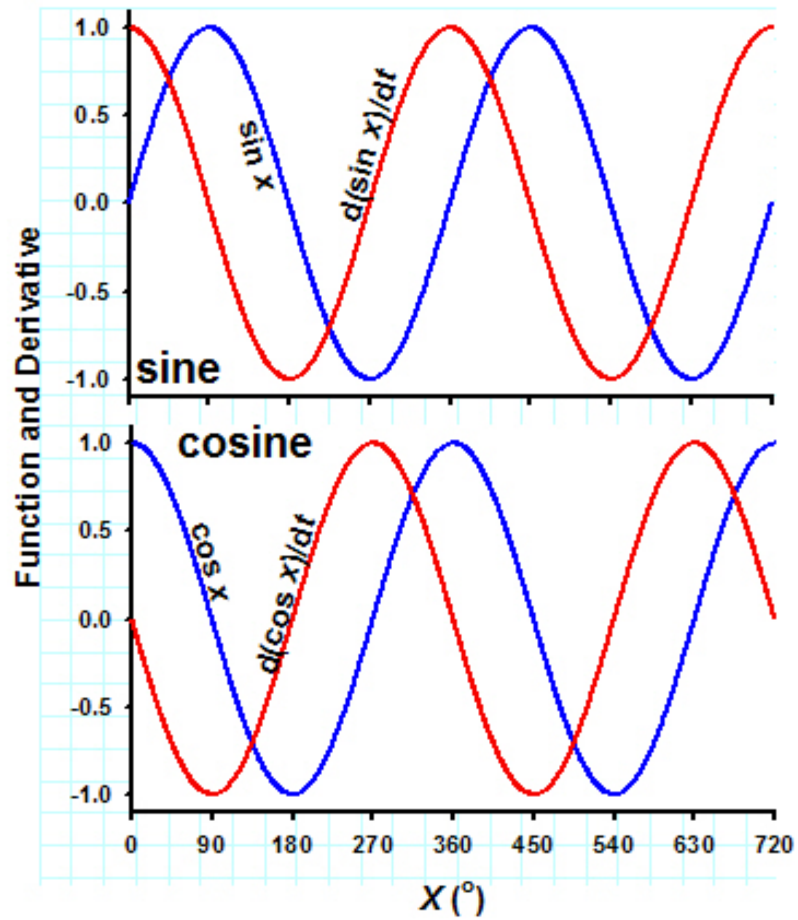
- Of course :

  $dx^0/dt =$ d constant $/dt = 0$



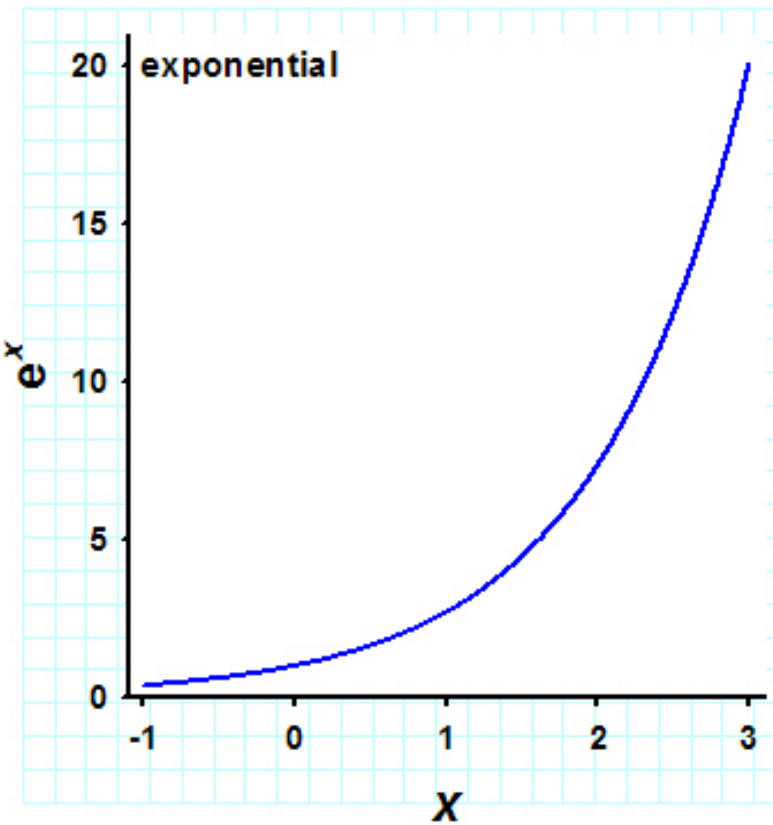
Derivative of sines & cosines

- $d(\sin x)/dx = \cos x$
- $d(\cos x)/dx = -\sin x$
- Derived as for powers, but using trig sum-of-angle formulas
- $d^2(\sin x)/dx^2 = -\sin x$
- $d^2(\cos x)/dx^2 = -\cos x$



Exponential Function

- $e^x = \exp\{x\} = (2.71828\ldots)^x$
- *e* is Euler's number, or the base of the natural logarithms
- $e^x * e^y = e^{x+y}$
- $(e^x)^n = e^{nx}$
- $de^x/dx = e^x$
- Natural logarithm is the inverse of exponential
- $\ln(e^x) = x$
- $\ln x + \ln y = \ln xy$
- $n\ln x = \ln x^n$
- $d(\ln x)/dx = 1/x$



Some Other Calculus Rules

- Derivative of a sum
  - $d(x + y)/dt = dx/dt + dy/dt$
- Derivative of a product

  $dxy/dt = x\, dy/dt + (dx/dt)\, y$
- Chain rule for a function of a function:

  $d\, f(g(t))/dt = (df/dg)*(dg/dt)$

  Example: $d(\exp\{-kx^2\})/dx = \exp\{-kx^2\}\,(-2kx)$ or $d(\sin kx) = k \cos kx$

Partial derivatives: Differentiate a function of several variables only by the specified variable, keeping all others constant

$$\text{Let } g(x, y, z) = xy^2z^3, \text{ then}$$

$$\frac{\partial \mathrm{g}}{\partial x} = y^2z^3, \quad \frac{\partial \mathrm{g}}{\partial y} = 2xyz^3, \text{ and}$$

$$\frac{\partial \mathrm{g}}{\partial x} = 3xy^2z^2.$$

Summary of Derivatives

- Derivatives represent rates of change of functions.
- Easily remembered recipes for calculating them from functions:
  - $d\, x^n/dx = nx^{n-1}$
  - $d(\cos x)/dx = -\sin x$; $d(\sin x)/dx = \cos x$
  - $d\, e^x/dx = e^x$; $d(\ln x)/dx = 1/x$
- Chain rule
- Rules for sums and products
- Partial derivatives

Integration is the opposite of differentiation—the anitderivative

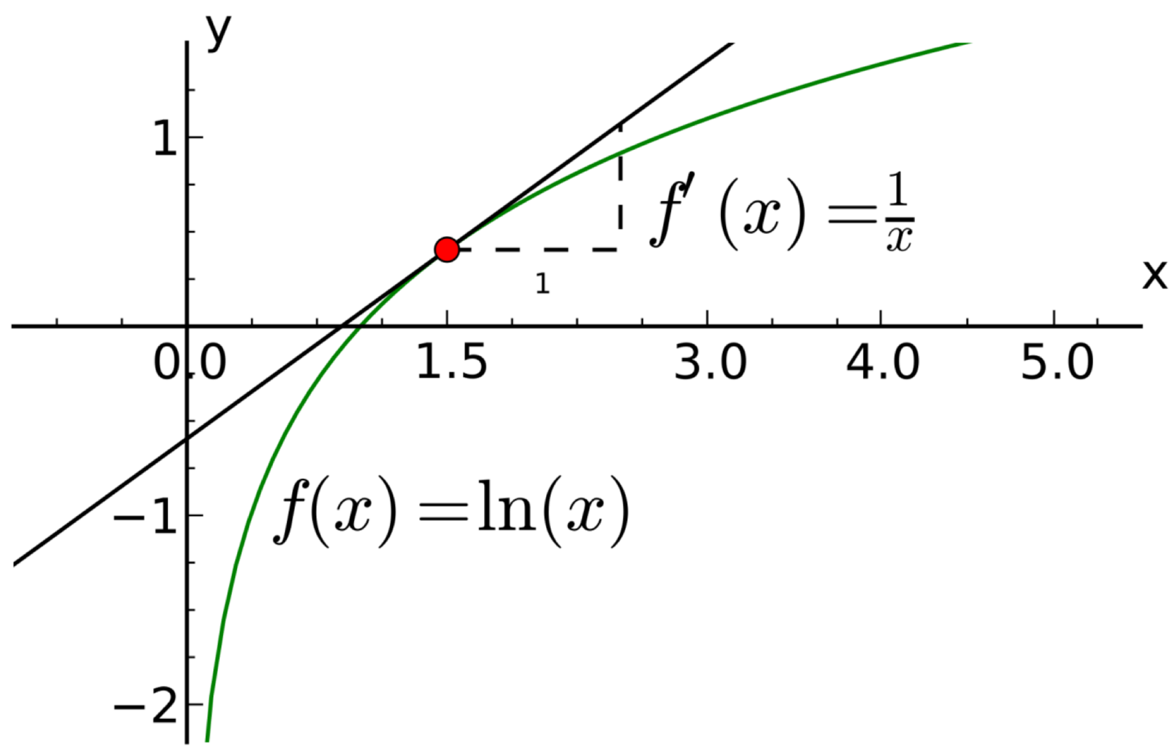
$$F(x) = \int \frac{dF(x)}{dx} dx + \text{constant, or}$$

$$f(x) = \frac{d}{dx}\left(\int f(x) dx + \text{constant} \right)$$

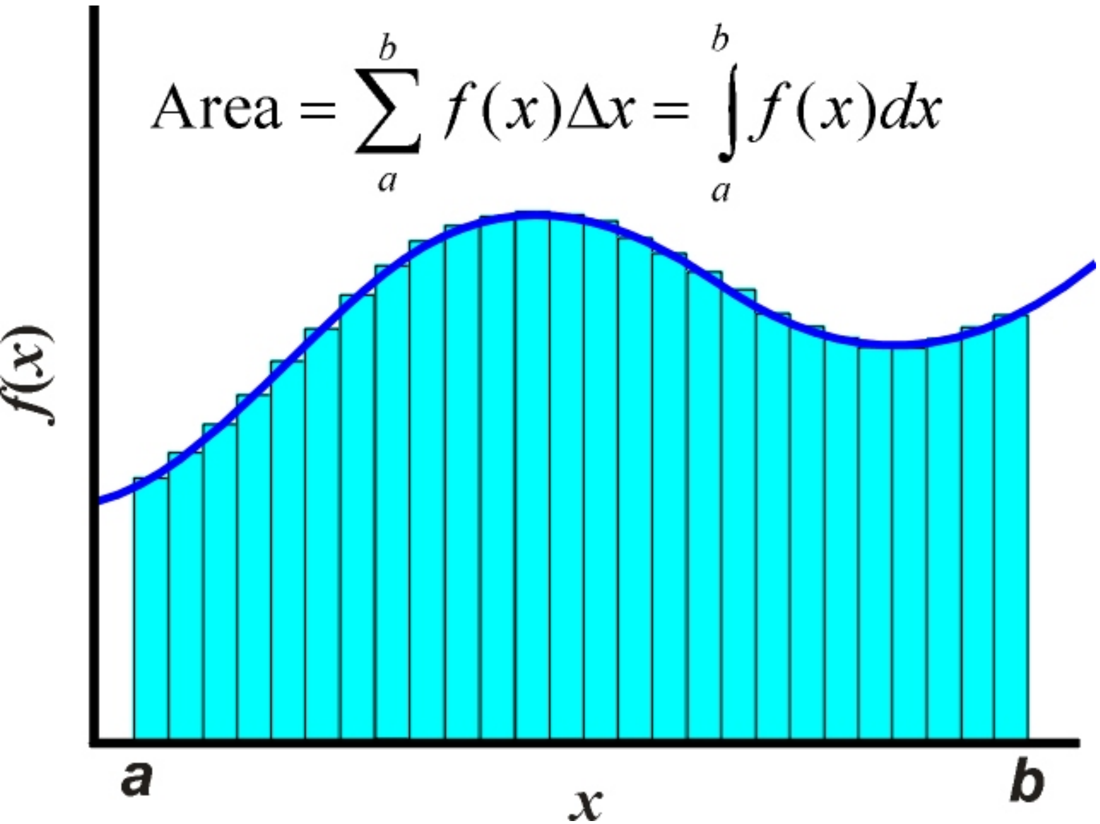
Where $f(x)$ is any function of x

Geometric Meaning of Derivative

- Geometrically, the derivative of a function f(x) can be interpreted as the slope of the curve of the mathematical function f(x) plotted as a function of x.
- The derivative can give you a precise instantaneous value for the rate of change of some physical parameters, such as speed, acceleration, force, etc.



Geometric Meaning of Integral



- Geometrically, the integral of the function f(x) over the range x=a to x=b gives the area under the curve between these two points.
- The integral gives you a mathematical way of drawing an infinite number of blocks and getting a precise analytical expression for the area.

We can turn the derivatives that we already know into integrals

$$\int x^n dx = \frac{1}{n+1}\int (n+1)x^n dx = \frac{x^{n+1}}{n+1} + \text{constant}$$

$$\int \sin x \; dx = -\cos x + \text{constant}$$

$$\int \cos x \; dx = \sin x + \text{constant}$$

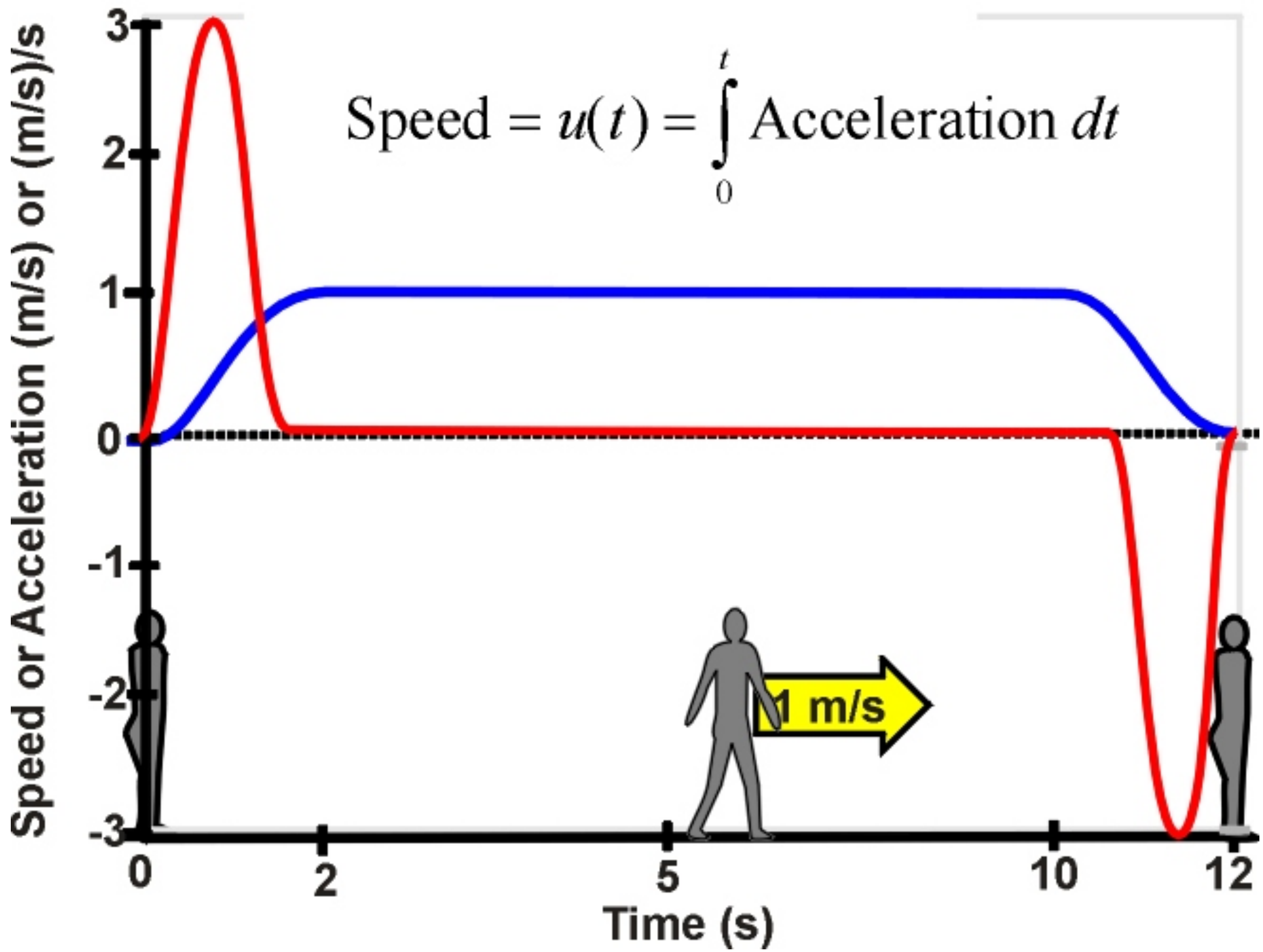
$$\int e^x dx = e^x + \text{constant}$$

$$\int \frac{dx}{x} = \ln x + \text{constant}$$

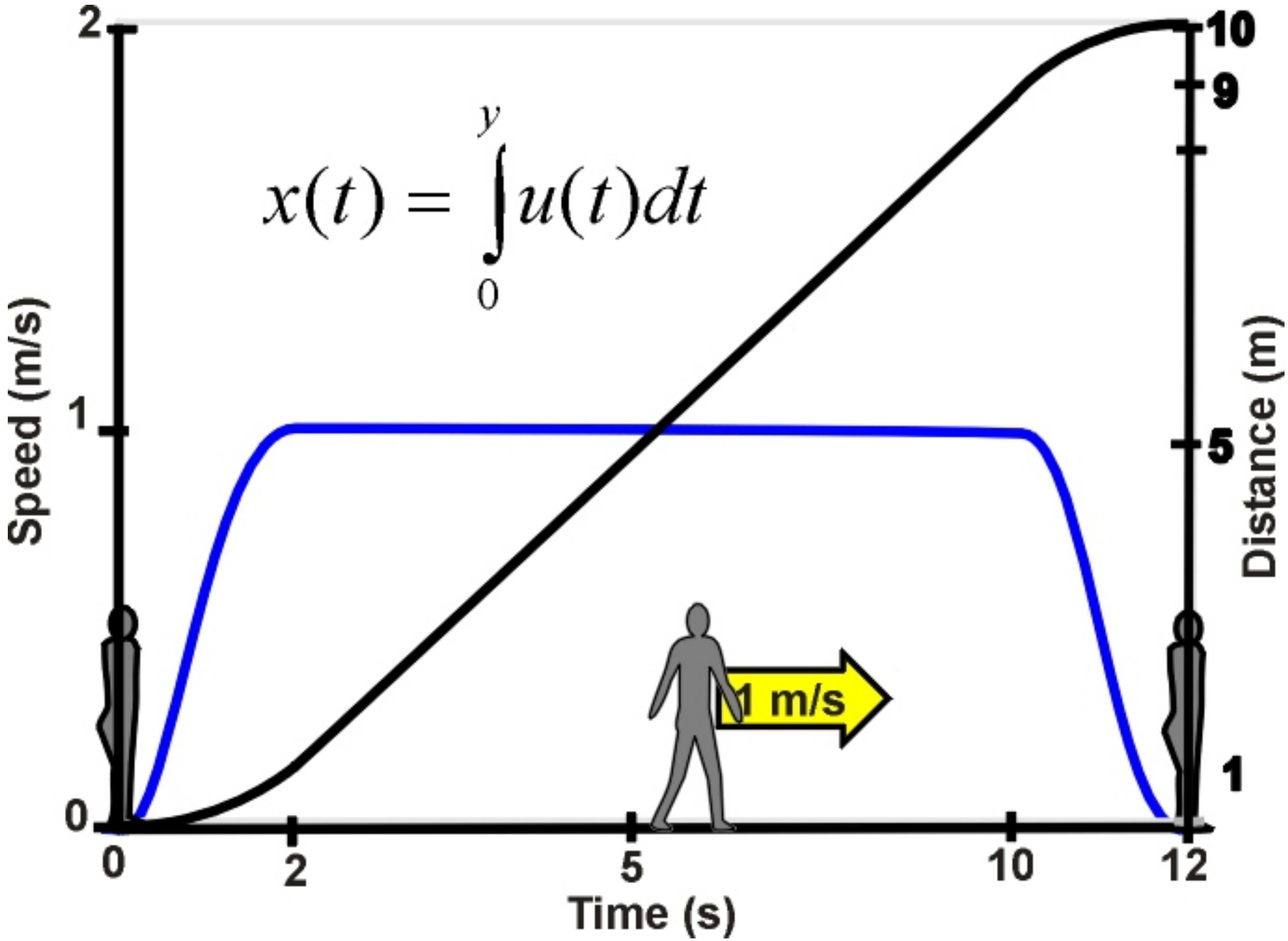
Basic Rules of Integration

1. $\int cf(x)dx = c\int f(x)dx$, c is a constant
2. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
3. $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x) = \int f(x)dx$
4. $\int_a^a f(x)dx = 0$
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
7. $\int_a^b c\,dx = c(b-a)$

Back to the Professor's Stroll: integrating his acceleration gets speed



Integrating his speed gets distance



Summary of Integration

- Integration is the reverse of differentiation
- Areas under curves
- Constants of integration determined by limits
- Standard formulas