Lecture 10 Rayleigh and Mie Scattering

Review of some scattering/absorption (extinction) parameters (see Lec 6&7 for details):

1. Scattering/absorption coefficient:

$$\beta_{s\lambda}, \beta_{a\lambda} \qquad Unit: m^{-1}$$

2. Mass scattering/absorption coefficient (Specific scattering/absorption cross-section)

$$k_{s\lambda}, k_{a\lambda} \qquad Unit: \frac{area}{mass}$$

$$\beta_{s\lambda} = \rho k_{s\lambda} \qquad \beta_{a\lambda} = \rho k_{a\lambda}$$

Where ρ is the density of the medium.

3. Extinction coefficient: $\beta_{e\lambda} = \beta_{s\lambda} + \beta_{a\lambda}$

$\beta_{e\lambda}$ is the rate of power attenuated per unit distance due to extinction.

4. Mass extinction coefficient $k_{e\lambda} = k_{a\lambda} + k_{s\lambda}$ and $\beta_{e\lambda} = \rho k_{e\lambda}$

$k_{e\lambda}$ has dimensions of area per mass. $k_{e\lambda}$ is also considered as *an extinction cross-section ($\sigma_{e\lambda}$, see definition below) per unit mass.*

Some new scattering/absorption (extinction) parameters:

1. Scattering/absorption cross section: $\sigma_{a\lambda}, \sigma_{s\lambda}$

Previously, the mass scattering/absorption coefficient was defined by finding the proportionality between the volume scattering/absorption coefficient ($\beta_{a\lambda}, \beta_{s\lambda}$) and the mass density ρ of the medium. Now the scattering/absorption cross section ($\sigma_{a\lambda}, \sigma_{s\lambda}$) is defined by relating the volume scattering/absorption coefficient ($\beta_{a\lambda}, \beta_{s\lambda}$) to the number density (concentration N_p) of particles within the medium:

$$\beta_{a\lambda} = N_p \sigma_{a\lambda} \ , \quad \beta_{s\lambda} = N_p \sigma_{s\lambda}$$

The constants of above proportionalities $\sigma_{a\lambda}, \sigma_{s\lambda}$ has dimensions of area (m^2). **$\boldsymbol{\sigma_{a\lambda}, \sigma_{s\lambda}}$ are referenced to a single particle, not a mass of medium as** $\beta_{s\lambda}, \beta_{a\lambda}$ **or** $k_{s\lambda}, k_{a\lambda}$.

2. Extinction cross-section: $\sigma_{e\lambda} = \sigma_{s\lambda} + \sigma_{a\lambda}$

The extinction cross-section $\boldsymbol{\sigma_{e\lambda}}$ can be thought of as the equivalent area of an opaque object blocking the same total amount of radiation. $\boldsymbol{\sigma_{e\lambda}}$ is different with the extinction coefficient $\boldsymbol{k_{e\lambda}}$ in that $\boldsymbol{\sigma_{e\lambda}}$ is referenced to a single particle while $\boldsymbol{k_{e\lambda}}$ is referenced to a unit mass. Since $k_{e\lambda}$ is an extinction cross per unit mass, we have:

$$\sigma_{e\lambda} = k_{e\lambda} m$$

where *m* is the mass per particle

3. Scattering and absorption efficiencies:

$$Q_{s\lambda} \equiv \frac{\sigma_{s\lambda}}{A} \qquad Q_{a\lambda} \equiv \frac{\sigma_{a\lambda}}{A}$$

Where A is the geometric cross-sectional area of the particle. In case of a spherical droplet, $A = \pi r^2$, where *r* is the radius.

4. Extinction efficiency: $Q_{e\lambda} = Q_{s\lambda} + Q_{a\lambda}$

   In the visible and IR band, a single cloud droplet has an extinction cross-section that is similar to, but not identical to its geometric cross-section *A*. It is possible that a particle can extinguish more radiation than its geometric cross-section would imply. In another word, $Q_{e\lambda}$ could be greater than 1!

Relative index of refraction:

$$m = \frac{N_2}{N_1}$$

Where N_1: complex refractive index of the particle; N_2: complex refractive index of the surrounding medium (which is the air for meteorology). Recall $N = n_r + i\, n_i$, *Note that* n_r governs the phase speed of propagation of a wave within the material; n_i governs absorption.

N_1 is usually taken to be equal to 1 for air, so m approximates to N_2.

N_2 depends on both the composition of the particles and on the wavelength λ.

Rayleigh Scattering:

Size parameter ($x = \frac{2\pi r}{\lambda}$) range: $x < 0.2$ $(or\ sometimes\ refered\ to\ as\ x \ll 1)$

Rayleigh Solution: to answer how much radiation a small particle scatters and/or absorbs in the atmosphere. The solution for Rayleigh extinction efficiency Q_e is (a complex number):

$$Q_e = 4xI_m\left\{\frac{m^2-1}{m^2+2}\left[1+\frac{x^2}{15}\left(\frac{m^2-1}{m^2+2}\right)\frac{m^4+27m^2+38}{2m^2+3}\right]\right\} + \frac{8}{3}x^4 R\left\{\left(\frac{m^2-1}{m^2+2}\right)^2\right\}$$

$$Q_s = \frac{8}{3}x^4\left|\frac{m^2-1}{m^2+2}\right|^2$$

Where I_m {} means taking the imaginery part, and *R*{} means taking the real part, *x* is size parameter, *m* is the relative index of refraction of the scattering particles.

Since $Q_a = Q_e - Q_s$, for sufficiently small x,

$$Q_a = 4xI_m\left\{\frac{m^2-1}{m^2+2}\right\}$$

Interpretation:

1) $Q_a \propto x$; $Q_s \propto x^4$, the absorption efficiency is proportional to size parameter *x*, while the scattering efficiency is proportional to x^4.
2) For sufficiently small x, assuming m has non-zero imaginary part: $Q_s \ll Q_a$ & $Q_a \approx Q_e$
3) The single scattering albedo $\omega = \frac{Q_s}{Q_e} \propto x^3$, is proportional to x^3.

Key facts about scattering and absorption in Rayleigh Regime:

1) $\sigma_s = Q_s A = \pi r^2 \cdot Q_{s,}\ \ \sigma_s \propto r^2 \cdot x^4 \sim \frac{r^6}{\lambda^4}$, scattering cross section is proportional to $\frac{r^6}{\lambda^4}$.
Therefore: A): if you have radiation of fixed wavelength λ and use it to illuminate two particles of radius $r_1 < r_2$, the larger particle will scatter the radiation more strongly by a factor of $\left({}^{r_2}/{}_{r_1}\right)^6$
——**This is of central importance of weather radar (larger particles produce much larger reflectivity than smaller particle).**
B): If you have a particle of fixed size and expose it to radiation with two different wavelengths $\lambda_1 < \lambda_2$, then it will scatter the shorter wavelength $\lambda_1$more strongly by a factor of $\left({}^{\lambda_2}/{}_{\lambda_1}\right)^4$.
——**This fact explains why the sky is blue (blue light has smaller wavelength than other colored lights in visible band).**

2) For sufficiently small particles with complex (not pure real) refractive index m, scattering is negligible and absorption is proportional to mass path only, independent on particle size.
——**This fact is relevant to passive microwave remote sensing of cloud water, where we don't need to consider particle size variations.**

$$since\ k_a = \frac{area}{mass} = \frac{\sigma_a}{mass} = \frac{Q_a \pi r^2}{\rho \cdot \left(\frac{4}{3}\right)\pi r^3}$$

$$= \frac{3Q_a}{4\rho r} = \frac{6\pi}{\rho\lambda} I_m \left\{\frac{m^2-1}{m^2+2}\right\}$$

$\Rightarrow\ k_a$ is independent on r.

Where r is the radius of the water droplet, ρ is the density of water

Mie Solution: (for a spheric particle)

$$Q_e = \frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)R(a_n+b_n)$$

$$Q_s = \frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)(|a_n|^2+|b_n|^2)$$

Where the coefficients a_n and b_n are Mie Scattering Coefficients and are functions of x and m. n is any integer.

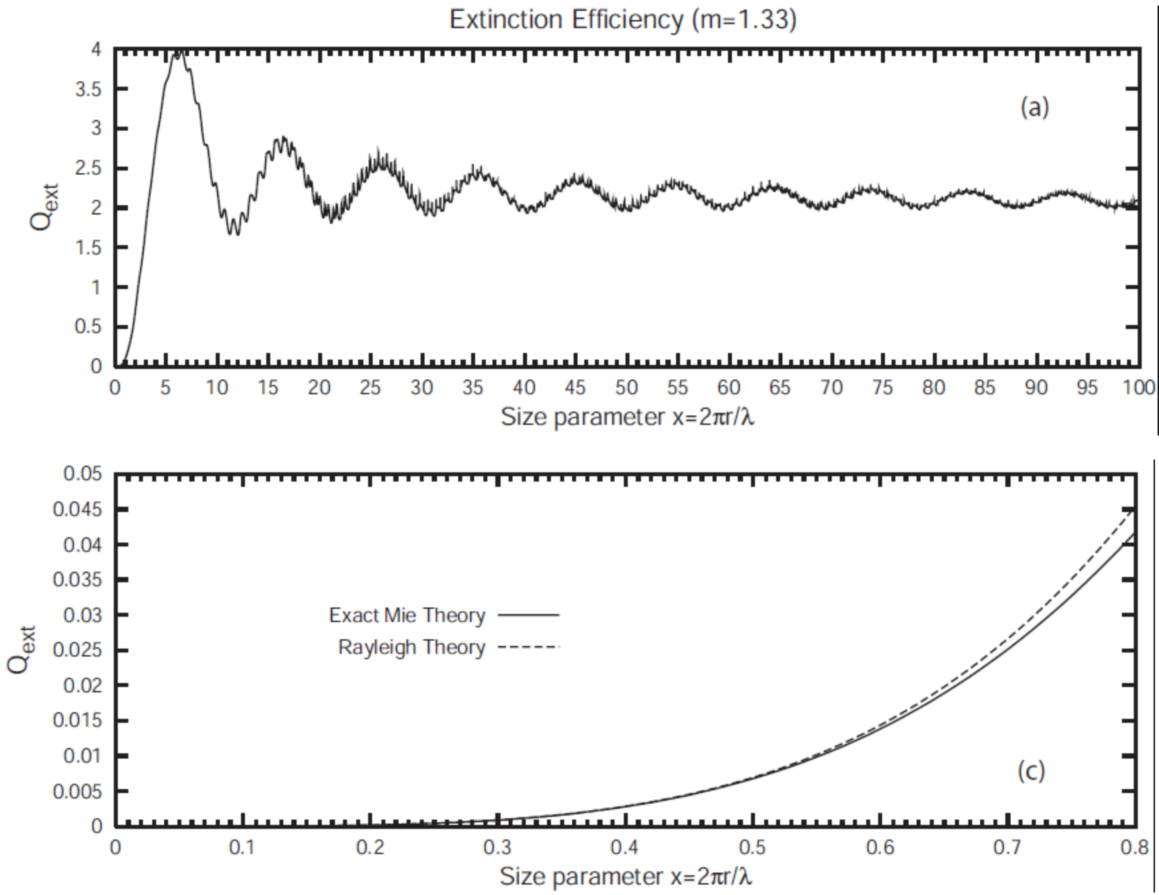

Fig. 12.4 from Petty's textbook: The extinction efficiency Q_e as a function of size parameter x for a non-absorbing sphere with relative refraction index *m*=1.33 for various ranges of *x*. (a) "Big picture" view showing that $Q_e \to 2$ as $x \to \infty$. (c) Detail for *x<0.8*, comparing the Rayleigh (small particle) approximation and exact Mie theory.

Explanation Petty's textbook Fig. 12.4:

a) —— Extinction efficiency Q_e increases as size parameter x increases up to $x = 6$, where $Q_e = 4$; after $x > 6$, $Q_e \to 2$ as $x \to \infty$ (extinction efficiency Q_e oscillates around 2 as size parameter x increases up to infinity)
b) ——Rayleigh Approximation is good up to $x = 0.6$ for a non-absorbing sphere with relative refraction index m=1.33.