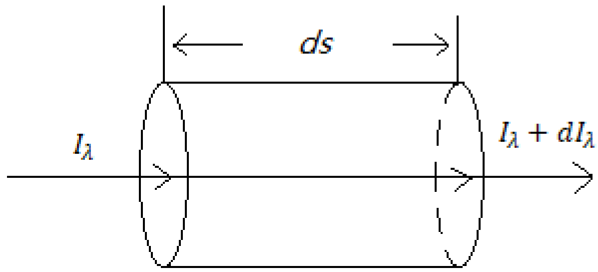
Lecture 8: The Radiative Transfer Equation

Supplemental Reading: Kidder & Vonder Haar's book (Chapter 3.3), Petty's textbook (ch11.2)

Concept: consider radiation incident on a differential volume of a medium (atmosphere or other material).



For the Radiative Transfer Equation, what do we want to get?

– The change in radiance I_λ as the radiation passes through the volume.

Q: If no material in the volume, will the radiance change?

A: No.

Which processes do we need to consider?

A: Radiation from the beam can be absorbed by the material.

B: Radiation can be emitted by the material.

C: Radiation can be scattered out of the beam into other direction.

D: Radiation from other directions can be scattered into the beam.

Therefore, the rate of change of radiance with distance, $\frac{dI_\lambda}{ds}$, consists of the above four terms:

$$\frac{dI_\lambda}{ds} = A + B + C + D$$

Depletion terms: A and C (remove radiation from the beam)

Source terms: B and D (add radiation to the beam)

Some scattering/absorption (extinction) parameters needed to derive the Radiative Transfer Equation (see Lec 6&7 for details):

1) Scattering/absorption coefficient:

$$\beta_{s\lambda}, \beta_{a\lambda} \qquad Unit: m^{-1}$$

2) Mass scattering/absorption coefficient (Specific scattering/absorption cross-section)

$$k_{s\lambda}, k_{a\lambda} \qquad Unit: \frac{area}{mass}$$

$$\beta_{s\lambda} = \rho k_{s\lambda} \qquad \beta_{a\lambda} = \rho k_{a\lambda}$$

Where ρ is the density of the medium.

3) Extinction coefficient: $\beta_{e\lambda} = \beta_{s\lambda} + \beta_{a\lambda}$

$\beta_{e\lambda}$ is the rate of power attenuated per unit distance due to extinction.

4) Mass extinction coefficient $k_{e\lambda} = k_{a\lambda} + k_{s\lambda}$ and $\beta_{e\lambda} = \rho k_{e\lambda}$

Derivation of the Radiative Transfer Equation:

1. From Lambert's Law with both absorption and scattering as extinction, we can get depletion terms A and C: $dI_{\lambda ext}$

$$\frac{dI_{\lambda ext}}{I_\lambda} = -\rho k_{e\lambda} ds$$

$$\Rightarrow dI_{\lambda ext} = -\rho k_{e\lambda} I_\lambda ds = -\beta_{e\lambda} I_\lambda ds$$

2. To get the emission term B, use Kirchhoff's Law under LTE, and Planck's function $B_\lambda(T)$:

The differential emissivity:

$$d\varepsilon_\lambda = \frac{dI_{\lambda emit}}{B_\lambda(T)}$$

According to Kirchhoff's Law:

$$da_\lambda = d\varepsilon_\lambda, \text{ where } a_\lambda \text{ is the absorptivity.}$$

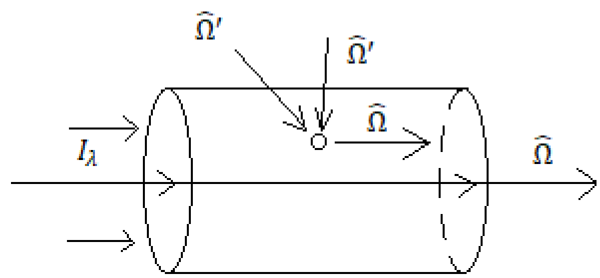
$$\Rightarrow dI_{\lambda emit} = B_\lambda(T) da_\lambda$$

From Lambert's Law, $da_\lambda = \rho k_{a\lambda} ds$, so:

$$I_{\lambda emit} = B_\lambda(T) \cdot \rho k_{a\lambda} ds = \beta_{a\lambda} B_\lambda(T) ds$$

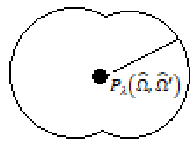
3. Term D: $dI_{\lambda scat}$

Radiation passing through our infinitesimal volume from any direction $\widehat{\Omega}'$ can potentially contribute scattered radiation in the direction of interest $\widehat{\Omega}$.



⇒ All directions must be considered!

Phases Function: to describe the angular distribution of scattered energy, we define a non-dimensional parameter called Phase function $P_\lambda(\widehat{\Omega}, \widehat{\Omega}')$, which corresponds to the fraction of radiation scattered by an individual particle from any direction $\widehat{\Omega}'$ into direction of interest $\widehat{\Omega}$.



Rayleigh Scattering

By definition, $\frac{1}{4\pi}\int_{4\pi} P_\lambda(\widehat{\Omega}, \widehat{\Omega}')\, d\Omega' = 1$, the phase function is normalized to unity (1).

Single Scattering albedo:

$$\omega_\lambda = \frac{k_{s\lambda}}{k_{e\lambda}} = \frac{\beta_{s\lambda}}{\beta_{e\lambda}}$$

which represents the fraction of radiation lost through extinction that is scattered out of the beam. Then,

$$1 - \omega_\lambda = \frac{k_{a\lambda}}{k_{e\lambda}}$$

because $$k_{a\lambda} + k_{s\lambda} = k_{e\lambda}$$

Therefore, term D should be:

$$dI_{\lambda scat} = \frac{\beta_{s\lambda}}{4\pi}\int_{4\pi} P_\lambda(\widehat{\Omega}, \widehat{\Omega}') I_\lambda(\widehat{\Omega}') d\Omega' ds$$

because just like what has been expressed in Lambert's Law, the fractional energy scattered is proportional to the mass traversed by the radiation. So,

$$dI_{\lambda scat} \sim I_\lambda \beta_{s\lambda} ds$$

(just like $dI_{\lambda ext} \sim I_\lambda \beta_{e\lambda} ds$)

But since we need to consider all directions, it must include integration over the phase function. So it becomes:

$$dI_{\lambda scat} = \frac{\beta_{s\lambda}}{4\pi}\int_{4\pi} P_\lambda(\widehat{\Omega}, \widehat{\Omega}') I_\lambda(\widehat{\Omega}') d\Omega' ds$$

4. Final radiative transfer equation:

$$dI_\lambda = dI_{\lambda ext} + dI_{\lambda emit} + dI_{\lambda scat}$$

$$= -\beta_{e\lambda} I_\lambda ds + \beta_{a\lambda} B_\lambda(T) ds + \frac{\beta_{s\lambda}}{4\pi}\int_{4\pi} P_\lambda(\widehat{\Omega}, \widehat{\Omega}') I_\lambda(\widehat{\Omega}') d\Omega' ds$$

Dividing through by $d\tau_\lambda = \beta_{e\lambda} ds$

(as defined in Lec. 7, τ_λ is optical path or thickness, $\tau_\lambda = \int_0^s \beta_{e\lambda} ds$)

$$\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda(\hat{\Omega}) + \frac{\beta_{a\lambda}}{\beta_{e\lambda}} B_\lambda(T) + \frac{\omega_\lambda}{4\pi} \int_{4\pi} P_\lambda(\hat{\Omega}, \hat{\Omega}') I_\lambda(\hat{\Omega}') d\Omega'$$

$$\frac{\beta_{a\lambda}}{\beta_{e\lambda}} = 1 - \omega_\lambda$$

Now we defined a source function:

$$J_\lambda(\hat{\Omega}) = (1 - \omega_\lambda) B_\lambda(T) + \frac{\omega_\lambda}{4\pi} \int_{4\pi} P_\lambda(\hat{\Omega}, \hat{\Omega}') I_\lambda(\hat{\Omega}') d\Omega'$$

⇒ Convenient Radiative Transfer Equation:

$$\frac{dI_\lambda(\hat{\Omega})}{d\tau_\lambda} = -I_\lambda(\hat{\Omega}) + J_\lambda(\hat{\Omega})$$

No-Scattering Equation:

According to Petty's textbook Fig. 12.1, when the size parameter x<0.002, it falls into the negligible scattering regime. For example, radiation in IR band passing through air molecules fits this condition. In this condition, scattering is negligible. Then:

single scattering albedo $\omega_\lambda = 0$

⇒ The radiative transfer equation becomes:

$$\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda + B_\lambda(T)$$

No-emission Equation:

At visible or near-IR band, the atmosphere doesn't emit significant amount of radiation, therefore, $B_\lambda(T)$ can be neglected. Then the radiative transfer equation becomes:

$$\frac{dI_\lambda(\hat{\Omega})}{d\tau} = -I_\lambda(\hat{\Omega}) + \frac{\omega_\lambda}{4\pi} \int_{4\pi} P_\lambda(\hat{\Omega}, \hat{\Omega}') I_\lambda(\hat{\Omega}') d\Omega'$$

Also, at visible band, it is acceptable to assume absorption in clouds is zero, ⇒ $\omega_\lambda = 1$. This is because neither liquid water nor water vapor absorbs much radiation in the visible band.