MAD 2104 Jan 29, 2015
Exam I and Key Prof. S. Hudson

1) Write out (in words) the converse of this sentence:

If it is wet in India, then it is dry in Pakistan.

2) Give the *precise* definition of $p \to q$ (of "if p then q"). I am not looking for a wordy discussion. Perhaps some kind of formula or chart would be better.

3) Compute (eg simplify) this formula: $(0101 \wedge 0110) \vee 0010$.

4) [10pts each] Prove or disprove. That means - answer each of these with True or False. Then explain and *justify* your reasoning. You can use well-known facts such as $\pi < 4$, the transitive property, or anything proved in class. You can assume the universal set is $U = R$, the real numbers. Be very careful to get the TF parts correct, otherwise I cannot give partial credit. But your proofs will count at least as much.

4a) $\forall x, x > \pi \to x > 4$.

4b) $\forall x \in Q, x^2 = 2 \to x < 10$.

4c) $\forall \epsilon > 0, \exists \delta > 0, \epsilon\delta = 2$.

4d) $\neg(p \oplus q) \equiv (p \to q)$

5) Steve would like to determine the relative salaries of three coworkers using two facts. First, he knows that if Fred is not the highest paid of the three, then Janice is. Second, he knows that if Janice is not the lowest paid, then Maggie is paid the most. Is it possible to determine the relative salaries of Fred, Maggie and Janice from what Steve knows ? If so, who is paid the most and who the least ? Explain your reasoning (for max credit, this should be a logical sequence of deductions, with clear reasons).

6) What rule of inference is used in this famous argument ? "All men are mortal. Socrates is a man. Therefore, Socrates is mortal." If you do not know the name of the rule, you can write it out instead (in the usual column format) for full credit.

7) [10pts total] Consider the proposition $\forall x, (x > 2 \to x > 3)$. It is true for some domains but not all.

7a) Give an example of a domain (a set of numbers, of course) for which it is false, with a brief explanation.

7b) Give a domain for which it is true. No explanation required.

Remarks and Answers: The average among the top 35 grades was 77 out of 100, which is very good, but not unusual for the first test. The 4 highest scores were all in the 91-92 range. The best results, 90% or better, were on problems 2, 3, 4a and 4d. The lowest results, 50% to 60%, were on problems 4b, 4c and 6.

You can use the scale on the syllabus to convert your numerical score on this exam into a letter grade. An 85, for example, is the low A-. It is fairly likely that the average on the next exams will be lower than 77 and that I will lower the scale for those exams.

1) If it is dry in Pakistan then it is wet in India.

2) I gave full credit for the truth table [see the text or lecture notes] with some sentence - almost *any* sentence - of explanation. For example "The $\rightarrow$ operator can be defined by its truth values for each case, as listed in this table." I also accepted formulas, if put into context, such as "It is logically equivalent to $\neg p \vee q$".

3) 0110 (and show work).

4a) False (this was worth 5 points). Set $x = 3.5$. This is a counterexample, because $3.5 > \pi$ but $3.5 \not> 4$ (this part was also worth 5 points).

Other examples such as $x = 4$ are also OK, but not $x = 3.14$. I gave credit for various other explanations if they included some specific number like 3.5, and at least a few (correct) words. No partial credit without "False". No credit for explanations that I could not follow.

4b) True. The obvious strategy is a *vacuous proof* such as this:

Proof: Fix $x \in Q$. If $x^2 = 2$ then either $x = \sqrt{2}$ or $x = -\sqrt{2}$, but neither is rational (by the theorem proved in class, and simple algebra). So, $x^2 = 2$ is False, which makes the implication True.

Unfortunately, some people tried to write a direct proof, which I did not expect. This is not automatically 100% wrong, but it makes the problem harder, since we have not learned much about square roots in this class. Usually, it did not go well. For example:

Since $x = \sqrt{2}$, we know that $x < 2$, because $\sqrt{y}$ is always less than y. So, $x < 2 < 10$.

Forgetting that $x = \sqrt{2}$ is impossible is not fatal, but there are other problems. They forgot to mention the case $x = -\sqrt{2}$). Also, $\sqrt{y}$ is not always less than y (try $y = 1/4$). There were other similar attempts, but I recall none that were 100% acceptable.

4c) True. Fix $\epsilon > 0$. Set $\delta = 2/\epsilon$ (since $\epsilon \neq 0$, this is OK). So, $\delta > 0$ and $\epsilon\delta = 2$. Done.

This proof includes the standard methods for handling $\forall$ and $\exists$. Some people seemed confused by the Greek letters (I used them here and in class because they are the standard choice in limit proofs). I allowed these people to rewrite the problem as $\forall x > 0, \exists y > 0, xy = 2$ which is equivalent.

4d) False. A truth table shows they are not LE. Include some comment. I accepted almost any comment, such as "No, this shows they have different truth values".

5) This is 1.2.33. Yes. F,M,J : Fred is paid the most and Janice the least - but on some versions of the exam the names were Fran, Maggie and Jose. This part was worth 5 points, but I usually gave no partial credit without it. For full credit, your explanation had to make perfect sense to me after 1-2 readings (which probably happened half the time). Here is one good student answer:

Maggie cannot be highest paid because that would contradict statement 1. So, Janice must be lowest paid, by statement 2. Since neither Maggie or Janice is highest paid, it must be Fred.

There were many many ‘explanations’ that did not work for me, even if all the sentences were true. For example:

Bad ex: If Fred is paid the most, then Maggie cannot be paid the least. So Janice is. FMJ. (Done).

There are a few problems with this. First, I do not see the reasoning behind ‘then Maggie cannot be paid the least’ (though it is true). Second, the student does not mention the possibility that Fred is not paid the most. The student may have worked out the possibilities correctly in his/her head, or on a table, but this explanation is not convincing. I had to label this kind of answer as ‘not clear’, without much credit.

6) This is 1.6.7. The answer in the back of the book has two rules, universal instantiation + modus ponens (and I gave decent partial credit for mentioning just one of these). These can be combined into one rule, universal modus ponens, see page 77. So, that is also an acceptable answer. I did not ask you to memorize the rule names, so I also accepted the p, q versions of these rules.

A few people interpreted ”Socrates is a man” as an implication, which seems quite a stretch to me, so I gave little or no credit for those answers.

7a) Sample answer: Let $U = R$, the real numbers. Then $x = 2.5$ is a counterexample to the statement.

7b) Sample answer: Let $U = (5, \infty)$, the real numbers bigger than 5. This contains no counterexamples (numbers between 2 and 3).

For 7a, you can use *any set* that contains some number in the interval $(2, 3]$, and for 7b any set that contains none of them, but $U = Z$ is not OK, since it contains 3. For 7a, it is OK to write the answer as $(2, 3]$ (for example), or in the equivalent form $\{x \in R : 2 < x \leq 3\}$, but I had to deduct points for bad notation such as

$2 < x \leq 3$ (this is a $p(x)$, not a set)

$\forall x \in R, 2 < x \leq 3$ (this is even worse, it is a false statement)

$\{2.1, 2.2, \ldots 2.99\} = (2, 3)$ (also false, though either side alone would be OK)

$Q(2, 3)$ (not sure what this means - it looks like ordered pair notation)

We will study sets more in Ch.2, and I hope the notation will improve then, but always try to be as precise as possible in this course.