MAD 2104 Feb 26, 2015
Exam 1 + Key Prof. S. Hudson

1) [30 pts] Misc short answer questions:

a) Suppose $A_j = [j+2, j+8) \subset R$ for $j = 1, 2, \ldots 5$. Compute $\cap_{j=1}^{5} A_j$.

b) $\sum_{j=1}^{3} \sum_{i=1}^{2} (3+j) =$

c) Show (briefly) that $\lceil x \rceil$ is not one-to-one from $R$ to $R$.

d) Find the symmetric difference of these two sets: $A = \{1, 3, 5, 7, 9\}$ and $B = \{4, 5, 6, 7\}$.

e) Show (briefly) that $f(x) = 17x + 18$ from $R$ to $R$ is onto.

2) [10 pts] Prove part of DeMorgan's Law; that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ using the usual paragraph style (as in Ch 2, Ex 10.).

3) [10 pts] Compute the Boolean product $A \bigodot B$. For one entry, at least, show ALL your work.

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

4) [10 pts] Find 3 different sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule (if you use formulas, make $a_1 = b_1 = c_1 = 3$, but probably $a_4 \neq b_4$ etc).

5) [10 pts] Give an example of a set $A$ and a function $f : A \to A$, that is onto but not one-to-one. For max credit, answer with a set and then a formula for $f$ (perhaps using $|n|$ and/or some polynomial and/or $\lfloor x \rfloor$). If your example is complicated, explain it.

6) [20 pts] True-False (you do not have to explain).

If $f : A \to B$ and $f(A) = B$ then $f$ is surjective (onto).

If $f : V \to U$ is 1-1 and $S \cup T \subset V$, then $f(S \cap T) = f(S) \cap f(T)$.

The set of all finite bit strings is countable.

$\forall x, y \in R, \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$

For all sets, $A \bigoplus B = (A \cap B) \cup \overline{A \cup B}$.

$\exists a, b \in N, |a^2 - b^3| = 1$.

The of all ordered triples $(a, b, c)$ of integers is countable.

The set of all infinite sequences of 0s and 1s is countable.

If a matrix $A$ has no zero entry ($a_{ij} = 0$) then neither does $A^2$.

The sixth term of the Fibonacci sequence is 31.

7) [10 pts] Choose ONE proof (if you use the back, leave a note here). Explain completely.

a) Prove by induction: for every positive integer $n$, 3 divides $n^3 + 5n$.

b) Prove by induction: $1^2 + 3^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ for $n \geq 0$.

c) $(0, 1)$ is not countable (use Cantor's diagonal argument).

**Remarks and Answers:** The average was 56 out of 100, based on the top 30 scores. This is low, of course, approx 15 to 20 points below Exam 1. The best scores were 82 and 71. The averages per problem did not vary much except for problem 3 (90%) and problem 5 (5%). Here is a rough scale for Exam 2:

A's - 68 to 100
B's - 58 to 67
C's - 48 to 57
D's - 38 to 47

You can estimate your current semester grade by averaging your two exam scores and using the scale below. For example, if you got an 82 and a 50, then your average is 66, which is a B-, almost a B. Averaging your two letter grades ought to give approx the same thing, but maybe less accurate (the scaling system is slightly subjective). For simplicity, I am not including HW grades yet, but if yours are unusual, you could do that. See me if you need help.

A's - 74 to 100
B's - 64 to 73
C's - 54 to 63
D's - 44 to 53

1) These were only 6 points each, so I did not give much partial credit. I gave a few points when the work was very clear so that the mistake was obvious - and fairly minor.

1a) $[7, 9)$. Common mistakes were to confuse $\cap$ with $\cup$ or even with $\Sigma$.

1b) 30. One simple method is to sum over j first, to get 15, and then over i. Since there is no 'i' in the summand, this means add 15 to itself.

1c) Since $\lceil 4.2 \rceil = \lceil 4.3 \rceil = 5$ but $4.2 \neq 4.3$, the function is not 1-1 (of course, there are many other equally good counterexamples). Since this is a 'show that' problem, you needed to include at least a few words for full credit.

1d) $A \oplus B = \{1, 3, 4, 6, 9\}$.

1e) ETS $\forall y \in R, \exists x \in R, 17x + 18 = y$. This is the definition of *onto* and it is [almost] required here, but the English version is also OK. Then, just show that you can solve this for $x$ (without running into trouble such as $x^2 = -1$ or $0 \cdot x = 1$, etc).

I sometimes gave a little credit for relevant facts such as 'it passes the horizontal line test' or 'the range equals the codomain' or 'it has an inverse function'. But each of these requires some justification. None represent much of a proof without that. No credit for 'it is increasing' or 'it is a polynomial' or 'it is continuous' or for xy tables.

2) This is in the text. If you didn't get full credit, compare your answer with that proof. Near the beginning you need to write something like this:

*Assume that* $x \in \overline{A \cap B}$. (see line 4 of Ex 10).

Near the end you need to write 'so $x \in \overline{A} \cup \overline{B}$'. Near the middle you need to mention De Morgan's law of logic. Do not mimic Exs 11 or 13 (these are not in the required style).

If you began with a phrase such as '$\overline{A \cap B}$ means elements that (etc)', you are being too wordy. It is difficult to convey the right ideas precisely this way.

3) The $c_{11}$ entry is $(1 \wedge 1) \vee (0 \wedge 0) = 1 \vee 0 = 1$, for example.

$$A \odot B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

4) This is 2.4.8. It is fairly easy with recursion, but be sure to include every part the RDs, such as 'for $n \geq 1$'. Otherwise, your answer may be virtually impossible to decipher. Here are some of the simplest answers. I think the first three are best:

Let $a_n = 2n + 1$ for $n \geq 1$ [the same sequence can be defined recursively using $a_n = a_{n-1} + 2$, but don't answer both ways].

Let $a_{n+2} = a_n + a_{n+1} - 1$ for $n \geq 1$, and let $a_1 = 3$, $a_2 = 5$.

Let $a_{n+2} = a_n a_{n+1} - 8$ for $n \geq 1$, and let $a_1 = 3$, $a_2 = 5$.

Let $a_{n+3} = a_n + 5$ for $n \geq 1$, and let $a_1 = 3$, $a_2 = 5$, $a_3 = 7$ [$a_n + 500$ works just as well. This kind of answer somewhat defeats the purpose, since it does not show where the 7 comes from, and is not so simple, but if it was presented clearly, I gave full credit for it].

Repeat 3, 5, 7, 3, 5, 7, 3, etc forever.

Let $a_n = 2(n-2)^3 + 5$ for $n \geq 1$ [by me, using some trial and error.]

A few people wrote *the odd integers greater than 1* which is roughly the same as the first answer above. It's slightly wrong because it defines a set, not a sequence. I gave more credit for *list (in order) the odd integers greater than 1*, since we can view a list as a sequence. Also, I can interpret that phrasing as a 'rule'.

5) This is similar to 2.3.20 - 21 and some problems from my older exams. It is not possible if $A$ is finite, so use an infinite set such as $A = N$ or $A = R$. It's very helpful to think about this kind of problem (giving examples) *before* the exam, since it may be hard to find them with the clock ticking. Examples are also useful with True-False, and with understanding math in general.

a) Let $A = N$ and $f(n) = |n - 2|$. Note $f(1) = f(3)$. This formula does not work with $A = R$, but it works with $A = [0, \infty)$.

b) Let $A = N$ and $f(n) = \lfloor n/2 \rfloor$. Note $f(1) = f(0)$. This formula does not work with $A = R$.

c) Let $A = R$ and $f(x) = x^3 - x$. Note $f(1) = f(0)$. The graph (or some Calculus thinking) shows this is onto. This formula does not work with $A = N$ or $A = Z$.

Note that $A$ matters. Many people gave some random-looking formula for $f$ but did not specify $A$ and I could not give credit. In general, it was difficult to give much partial credit on this problem. Answering with an arrow diagram might be easier (try it!) but remember that $A$ must be the same on both sides, and infinite, so any diagram would need 'dot-dot-dot's.

6) TTTFF TTFFF. Many people thought the 9th one was True, but there are counterexamples with negative entries such as

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{so that} \quad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$