MAD 2104 Apr 2, 2015
Exam 3 Prof. S. Hudson

The questions are worth 10 points each, unless labeled. Simplify answers such as $P(5,3)$, but not answers such as 6!.

1) [5 pts each] a) How many bit strings of length 8 contain at least six 1's ?

1b) How many permutations of the letters FLORIDA contain the string LAF ?

1c) How many numbers must be chosen from the set $\{1, 2, 3, 4, 5, 6\}$ to be sure some pair of the chosen numbers adds up 7?

2) Prove by induction that $\sum_{k=0}^{n} k = n(n+1)/2$ for all $n \geq 1$. As usual, a large part of your grade will be based on correct organization and full explanation.

3) [5 points each] Let $A = \{0, 1, 2\}$. If any part of this problem is impossible, explain why.

a) Give an example of a relation R on A, that is symmetric but not transitive. You may describe R in words, or by a clearly-labeled matrix, or with a (di)graph.

b) Give an example of a relation S on A, that is transitive but not reflexive.

c) Give an example of a relation T on A, that is neither symmetric nor antisymmetric.

4) [10 + 5pts] Johnny tried to represent an equivalence relation R on the set $A = \{a, b, c, d, f\}$ with the graph below. It has 4 edges (the dashes), from a to b, b to c, etc. It is OK that he omitted the loops and arrowheads, since those are 'understood'. But he drew one edge in the wrong place. Which edge would you remove from his graph ? Which missing edge would you insert ?

a – b – c – d – f

4b) After you have corrected his graph, find the equivalence class $[c]$.

5) [20 points] Answer True or False:

If S is finite, no function $f : P(S) \to S$ can be 1-1.

$r(7, 2) = 7$

The relation R on the integers defined by $a < b$ is a partial order.

A relation R on a finite set A is symmetric if and only if M_R is a symmetric matrix.

If $|A| > 3$ then there are more symmetric relations on A than reflexive ones.

6) [8+7 pts] Find the number of edges in each of these special graphs. Show work (hopefully not just a drawing) and label your answers clearly.

a) Q_5

b) K_7

7) Choose ONE proof, and circle it. As usual, explain fully. If you answer elsewhere, leave a note (same for the Bonus, etc).

a) Show that $r(3,3) > 5$ as done in class (do not use the formula $r(3,3) = 6$).

b) State the binomial theorem and give a combinatorial proof of it.

c) Use Strong Induction to prove that every integer $n > 1$ can be factored into primes.

Bonus): Explain why, at every party with 100 people, there is a pair of people who know the same number of other people.

Remarks and Answers: The average among the top 22 was approx 55 out of 100. This is clearly lower than Exam 2, especially since fewer good scores were averaged this time. The high scores were 93 and 84. The average score per problem did not vary much except on # 4 (22%). Here is a rough scale for Exam 3:

A's - 66 to 100
B's - 56 to 65
C's - 46 to 55
D's - 36 to 45

You can estimate your current semester grade by averaging your three exam scores and using the scale below. See me if you need help.

A's - 72 to 100
B's - 62 to 71
C's - 52 to 61
D's - 42 to 51

1) 37, 5!, 4.

2) See the text or lectures. Remember that a proof must include *words*. I think 5 or 10 is a bare minimum here, but you might replace a few with abbreviations. It would be a good idea to use as many as you see in the textbook examples. The second half of this proof is mainly a calculation, so that part needs fewer words.

A few people, who did not seem totally lost, tried to write this without $\sum$, which is impossible. The $\sum$ is one of the main characters in this story.

3) Each part has many possible answers. I accepted them in any legible form. Here are some that are fairly easy to type:

a) $|a - b| = 1$, or $ab = 0$, etc

b) $a < b$, or $\emptyset$, etc. A common mistake with transitive: if you include, for example, the pairs $\{(1, 0),\ (0, 1)\}$ into R, you must also include $\{(0, 0),\ (1, 1)\}$.

c) $R = \{(1, 0),\ (2, 0),\ (0, 2)\}$. This is not symmetric without $(0, 1)$. This is not anti-symmetric because 0 and 2 are related in both orders but are not equal. In terms of M_R, there are two opposing 1's across the diagonal from each other [draw it to see this].

4) Remove the edge from c to d. Insert an edge from a to c. This partitions A into $[c] = \{a, b, c\}$ and $[d] = \{d, f\}$. So, it is clear that the new R is an EqR. It is also OK to replace the bc edge by a cf edge, but I don't see any other ways to do it.

5) TTFTF. The results were fairly bad. See me about these, if needed.

6a) 80. 6b) 21.

7) See the text or lecture notes. Most people chose (a), followed by (b). None are hard.

For 7a) the main idea is a picture (of K_5 basically, but with half the edges "dashed"). Include a few sentences about friends and the lack of triangles. You do not need the PHP for this. For 7b), discuss strings of length n with exactly k x's, etc. Do NOT set $n = 2$, which is just one example. 7c) Show that you know Strong Induction by using "the usual template". Include the Strong IH, eg a sentence such as "Assume every integer from 2 to n can be factored into primes".

B) [brief answer; see me for more. Also, this is an exercise in the text.] Suppose person n knows k_n other people, $0 \leq k_n \leq 99$. We cannot apply the PHP yet (because $100 \not> 100$) which makes this example a bit harder than average.

CASE 1: Suppose someone there knows 0 other people. Then $\forall n, 0 \leq k_n \leq 98$ (why?). Now we can use PHP $(100 > 99)$, to say two people have the same k_n.

CASE 2: Similar. Left to you.