MAD 2104 Oct 13, 2011
Quiz 4 Key Prof. S. Hudson

1) [30pts] Show that the union of two countable sets is always countable. As usual, a large part of your grade will be based on your explanation (use enough words!).

2) [20pts] How many bit strings are there of length 5 or less ?

3) [20pts] Compute the Boolean product $A \odot B$, where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

4) [30pt] Choose ONE proof. Use induction. You may answer on the back.

a) Prove for every positive integer n, that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.

b) Prove for every positive integer n, that 3 divides $n^3 - n$.

Remarks and Answers: The average of the top 20 grades was approx 74 / 100. The lowest grades were mostly on Problem 1, but the rest were OK on average. The scale is

A's 82 - 100
B's 72 - 81
C's 62 - 71
D's 52 - 61

As on Quiz 3, I have estimated your semester grade in the upper right corner. But this time, I added your best 3 out of 4 quiz grades, so far. The average for this was about 240 / 300, which is (of course) higher than before. I used a scale of A's 270-300, B's 240-270, etc. Because of the change in the calculations (eg now dropping the lowest grade), the new estimate may seem inconsistent with the one on Quiz 3. Please check all this, and see me if anything looks wrong to you.

1) There weren't many good answers to this one. The main steps should be something like this;

Countable infinite sets can be listed; let $A = \{a_1, a_2, \ldots\}$ *and* $B = \{b_1, b_2, \ldots\}$ *be two countable sets* (ideally, the cases where A or B is finite should be handled separately, but these cases are easy, and not-very-interesting, and left to you). *Then* $A \cup B$ *can be listed too;* $A \cup B = \{a_1, b_1, a_2, b_2, \ldots\}$*, so it is countable.*

I gave little or no partial credit for giving examples, or for only doing the case of two finite sets. It is possible to base a proof on a zig-zag path between the elements of A and B (though few people tried this) or on a formula for a 1-1 corr (very few people tried this, and it seems harder). This was exercise 2.4.40, and it was done in class.

2) $2^5 + 2^4 + 2^3 + 2^2 + 2^1 = 62$ (simplification not required). Before the Quiz, I announced that you should not count the empty bit string (mainly to preempt questions), but if you did this anyway, and got 63, I gave full credit. See exercise 5.1.12.

3) This is a slight variation of Example 10 from 3.8. Some fairly bad methods, such as computing the normal (non-Boolean) product, gave answers that were nearly correct. So unless you showed some work, I could not give much partial credit for a close answer.

$$A \odot B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

4a) is similar to Example 1 in Ch4.1, while 4b) is Example 8. Most answers were mostly OK, but a common problem was setting up the induction step properly. See page 273, for example, and study the 3 sentences after "INDUCTIVE STEP" carefully. Your proof should contain similar wording (and in my opinion, the author should also have mentioned that $k \geq 1$, as he did in his other examples).

Another common problem was to treat $P(k)$ as a number, rather than a statement. For example, avoid notation like $P(1) = 0$ or statements like "3 divides P(k)". This kind of notation might make sense in Calculus, but not in an induction proof.