MAD 2104 June 17, 2010
Final Exam and Key Prof. S. Hudson

1) How many ways can we arrange 5 men and 5 women in a row, if the men and women must alternate ?

2) Given the formula for the matrix A below, prove the one for A^n using induction. The f_n are the Fibonacci numbers [0, 1, 1, 2, 3, 5, etc] and you can use any well-known formula about them in your proof.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

3) [7 points each] Recall that N is the set of non-negative integers. Give an example, using a formula if possible, of a function $f : N \to N$ that is:

a) one-to-one but not onto.

b) onto, but not one-to-one (a different function from part a), of course).

4) [8 points each] Let $A_i = \{1, 2, 3, 4, \ldots i\}$ for $i = 1, 2, 3, \ldots$. Find these:

a) $\cap_{i=2}^{n} A_i$

b) $\cup_{i=3}^{n} A_i$

5) [20 points] Answer True or False:

$\forall x[P(x) \wedge Q(x)]$ is logically equivalent to $\forall x P(x) \wedge \forall x Q(x)$

$\exists x[P(x) \wedge Q(x)]$ is logically equivalent to $\exists x P(x) \wedge \exists x Q(x)$

If R is an equivalence relation, then $R^2 = R$.

If G is a digraph and $\sum_{v \in V} \deg^-(v)$ is odd, then $\sum_{v \in V} \deg^+(v)$ must be odd.

In Boolean algebra, $x(y + z) = xy + xz$.

6) Determine whether the two graphs below are isomorphic. If so, give an isomorphism (but you don't have to prove it is). If not, explain that with some isomorphism invariant.

7) What is the value of the postfix expression 7 2 3 * - 4 ↑ 9 3 / + ? (give a number)

8) Use a K-map to minimize this expression, $\overline{x}y\overline{z} + x\overline{y}\overline{z} + xy\overline{z} + \overline{x}\overline{y}\overline{z}$. [Check that you can read this clearly: there should be 8 letters over-lined in my formula.]

Bonus [maybe a hard 5 points]: How many bit strings of length 8 contain the string 10, but do not contain the string 11 ?

Remarks and Answers: The average of the top 20 grades was 67/100, which is normal for a quiz, and fairly good for a final exam. The results were generally better on the recent topics than the earlier ones (sets, induction, counting). There is no need to scale the final (but it would be similar to the one on the syllabus), and I have not yet set the scale for the semester averages.

1) [Avg 4/10] $2(5!)^2$ The most common mistake was to add instead of multiply, but this involves a sequence of decisions, so use the product rule.

2) [Avg 5/10] I won't write out the whole proof here, but the key part of the induction step is:

$$A^{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{pmatrix}$$

3) [Avg 11/14] a) $f(n) = n+1$. b) $f(n) = |n-1|$. There are many other answers, especially for a).

4) [Avg 8/16] a) $A_2 = \{1, 2\}$ (the smallest one), b) $A_n = \{1, 2, \ldots n\}$ (the largest one). Note: choosing the smallest/largest doesn't always work.

5) [Avg 16/20] TFTTT

6) [Avg 9/10] Yes. Match abcde with jlkmn in that order (but there may be other isomorphisms).

7) [Avg 6/10] 4 (start with 7-(2*3)=1, etc

8) [Avg 9/10] Draw a 2×8 grid. The 1's should form a 2×2 square [or maybe a 1×4 line] giving $F = \overline{z}$.

Bonus) [Avg 0.5/5] 53. Use recursion. One way to do that is to ignore the 10 rule, which leads to the Fibonacci number 55. Then subtract off 2 (the number of these that do not contain 10).