MAD 2104 Dec 8, 2011
Final Exam Prof. S. Hudson

1) [10 pts] How many bit strings of length 7 contain at least five 1's ?

2) [10 pts] Prove ONE, in the usual paragraph style, including the phrase $x \in$ more than once. Avoid shortcuts, such as quoting stronger theorems about sets (see me, if in doubt about this).

$\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

If $\overline{A} \subseteq B$, then $\overline{B} \subseteq A$.

3) [5 points each] Let $A = \{0, 1, 2\}$.

a) Give an example of a relation R on A, that is reflexive but not symmetric. You may describe R in words, or by a clearly-labeled matrix, or (di)graph.

b) Give an example of a different relation R on A, that is transitive but not an equivalence relation. If either part of this problem is impossible, explain why.

4) [10 points] Find a Boolean expression in sum-of-minterm form (eg DNF) for the function, $F(x, y, z) = 1$ if and only if $x + y = 0$.

5) [15 points] Answer True or False:

If $f : A \to A$ is onto, it must also be one-to-one.

The wheel graph W_n for $n \geq 3$ is never bipartite.

The complete graph K_n for $n \geq 4$ always has an even number of edges.

The number of different Boolean functions of degree n is always at least 4^n (for $n \geq 1$).

The terms of the Fibonacci sequence alternate between odd and even forever.

6) [10 pts] Determine whether the two graphs [below, or on the board] are isomorphic. If so, give an isomorphism f. If not, explain that with some isomorphism invariant.

7) [5 pts] In which order are the vertices of this ordered rooted tree [below or on the board] visited using a postorder traversal?

8) [10 pts] How many paths of length 6 are there in the graph from a to d? The following calculation may help a little.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 3 & 3 & 0 \\ 3 & 0 & 0 & 3 \\ 3 & 0 & 0 & 3 \\ 0 & 3 & 3 & 0 \end{pmatrix}$$

9) [10 pts] Use a K-map to minimize this expression, $x\overline{yz} + \overline{xyz} + \overline{x}y\overline{z} + \overline{xy}z + x\overline{y}z$. Check that you can read this clearly: there should be 10 letters over-lined, and $\overline{yz}$ means $\overline{y} \cdot \overline{z}$. If you can do this with QM (instead of a K-map), this problem is worth +2 points extra credit. Please don't do it both ways.

10) [5 pts] Define $f : N \to Z$ recursively by $f(0) = 1$, and $f(n) = -nf(n-1)$ for $n \geq 1$. Find $S = f^{-1}(N)$ (eg, the pre-image; describe S in words, or perhaps by a formula).

11) [5 pts] Prove that Z is countable. You may include a diagram, a matrix, arrows, lists, or a table (not all these seem like very good ideas) but include enough words to explain your thinking.

Bonus): Explain why, at every party with 100 people, there is a pair of people who know the same number of other people.

Remarks and Answers: The average was about 66 / 100, which is fairly normal. The results were poor on problems 8 and 11, and very poor on 10, but pretty good otherwise. Fortunately, the worst results were on 5 point problems. I have not yet set the scale for the semester grades, but don't expect it to differ much from the previously announced scales.

1) $C(7,5) + C(7,6) + C(7,6) = 21 + 7 + 1 = 29$

2) Most people chose the first one. The proof should begin with something very close to this;

Assume $x \in \overline{A \cup B}$. We will show that $x \in \overline{A} \cap \overline{B}$. (Etc)

3a) There are many good answers, such as $R = \{(0,0), (1,1), (2,2), (0,1)\}$. It is probably easier to draw a digraph (with 3 loops!) or a matrix, but my list is OK too and easier to type.

3b) $R = \{(0,1)\}$ is one good example.

4) $F = \overline{xy}z + \overline{xyz}$.

5) FTFTF; the 4th is OK because $4^n = 2^{2n} \leq 2^{2^n}$ = the number of functions.

6) Yes. Let $f(u_1) = v_1$, $f(u_2) = v_3$, etc.

Note that an isomorphism is a *function* from V_1 to V_2, and you should use notation to match. I did accept abbreviations using arrows or columns, and gave lots of partial credit for bad notation, such as $u_1 = v_1$. There is no reason to mention paths, or the number of edges, etc.

7) d, f, g, e, b, c, a

8) 18; compute $A^6 = (A^3)^2$ and look in the upper right corner; 18. You don't have to

compute all 16 entries of A^6, of course. But -

There is a mistake in the Hint; every 3 should be a 4. So, the correct answer is actually 32. I gave full credit for either 32 or 18 (or other answers derived from the flawed hint). No partial credit for other numbers without clear reasoning / work behind them.

9) $\overline{y} + \overline{xz}$

10) S is the set of non-negative even numbers (in the domain). For example, $f(2) = 2 \in N$, but $f(3) = -6 \notin N$. So $2 \in S$ but not 3.

11) Probably the simplest proof is to list the set, $Z = \{0, 1, -1, 2, -2, \ldots\}$ (with some explanation).

Bonus) There are two cases. Case A: Suppose everyone knows at least one other person, so the 'number known' ranges from 1 to 99. These 99 possibilities are shared by 100 people. By the PHP, at least two have the same number known. Case B: Suppose someone there knows no one. Then nobody there knows 99 others. So, the 'number known' ranges from 0 to 98. By reasoning similar to Case A, with PHP, two people share a number. Done.

Many people did Case A correctly and got 3 points, but nobody mentioned Case B. We went over this HW exercise in class, but it IS a tricky one.