MAD 2104 — April 30, 2015
Final Exam and Key — Prof. S. Hudson

1) Simplify as far as possible: $((A \cap B) \cup (\bar{A} \cap B)) \cap (A \cap B)$.

2) Let $\Sigma = \{a, b, c\}$ be an alphabet.

2a) How many words in Σ^4, such as $cacb$, contain at least one b ?

2b) Let $F = \Sigma^*$ be the set of all such words of finite length, such as $accbaccb$. Let I be the set of all words (or "sequences" actually) of infinite length, such as $accbaccba\cdots$. Do F and I have the same cardinality? Justify briefly, perhaps with a 1-1 correspondence or by discussing both cardinalities separately. You are not expected to repeat the proof of Cantor's theorem, for example, but you can refer to such a proof.

3) State the definition of $f : A \to B$ is *one-to-one*.

4) Let G be a simple graph with vertices $\{a, b, c, d, e\}$, with the adjacency matrix M given below. Draw G (or ask me to do it, but at a cost of about 4 points).

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

4a) Does G have an Euler path ? Justify your answer to this and the other parts of this problem below.

4b) Is G connected ?

4c) Is G planar ?

4d) Is G bipartite ?

4e) Is G isomorphic to W_4 (the Wheel with 5 vertices)?

5) Define a set S recursively by

$5 \in S$ and $9 \in S$ and
if $x, y \in S$ then $x + y - 2 \in S$.

Is $15 \in S$? Justify.

6) Use a binary tree to systematically find a prefix code for the letters a, b, c, d, e, f such that $a = 100$. Include 5 formulas like $b = 10110$ (but this formula might be wrong) in your answer. Remember that a prefix code is NOT about prefix notation.

7) Let Σ^5 be the bit strings of length 5. For $x, y \in \Sigma^5$, let $(x, y) \in R$ if they agree in the first 3 digits. For example, $(11001, 11011) \in R$.

(1a) Is R transitive on Σ^5 ?

(1b) Is it anti-symmetric ?

(1c) If R is an equivalence relation, how large are the equivalence classes ? For example, compute $|[10100]|$ and discuss briefly the other classes. If it is not an equivalence relation, prove that it is not.

8) [10 pts] How many different Boolean functions are there of degree 3 ?

9) [10pts] Draw a binary rooted tree T for the Polish notation expression $+3 \div yx$. Use your tree to convert it to Reverse Polish.

10) [10 pts] Perform the QM method on the expression $F(x, y, z) = xyz + x\bar{y}z + xy\bar{z} + \bar{x}\bar{y}\bar{z}$ to simplify F. This should be very quick, and you can skip "Stage 2" (about discarding some terms from your answer). Use the style and notation from the book or lectures. For partial credit, approx 50%, you can use a K-map instead. It is OK to use a K-map to check your answer, no penalty for that.

Bonus) How many ways can we put 3 identical eggs into 4 distinct baskets A, B, C, D such that either A is empty or D holds exactly one egg?

Remember - you should have justified most of your answers, especially the Yes/No answers, and shown some work for the rest. Have a great summer !

Remarks and Answers: This exam had approx 17 short problems, mostly for 5 points each. The average was 70% (good), with high scores of 98 and 90. The results were slightly better, about 77%, on the recent material in problems 8, 9 and 10. The worst scores were on 2 and 3 (just under 40%) and best on 5 (90%) and perhaps 4 (80%). As usual, I do not scale the final separately, but will scale the combined semester grades instead.

1) $A \cap B$

2a) $3^4 - 2^4 = 65$. Other methods seem much harder; only one succeeded.

2b) No, F is a countable union of finite sets (the set of words of length j) so it is countable. But I is like R, with letters replacing digits, and is not countable, by Cantor's argument.

Since 2b) is harder than 1) or 2a), I didn't insist on a full proof, and gave partial credit for anything close to this. I did not give much for answers like "No, there is no 1-1 correspondence from F to I".

3) It means $\forall x, y \in A(f(x) = f(y) \rightarrow x = y)$. I accepted the contrapositive and/or

less formal English versions, such as: Two distinct elements of the domain A cannot be assigned the same element $b \in B$. Many people had at least a vague idea of the meaning, and gave examples, but I could not give credit for that. You need the actual definition to write even very basic proofs about 1-1 (and most other discrete math vocabulary).

The results on this fairly easy problem were not very good. Many people gave the definition of *function*, or *onto*, or answers too vague to be understood (see 4b for a discussion).

4) I did not grade the drawing separately because you need it for the other parts. The graph is isomorphic to $K_{3,2}$. Nobody commented on that, but this fact could be used to explain 4a through 4d. Generally, I gave approx half credit on each part for a correct Yes or No, with full credit for that and a good justification.

4a) Yes, because exactly two vertices (b and e) have odd degree. This reasoning assumes G is connected, but I let that go since we get to that in (4b). Of course, it is OK to write out an Euler path instead.

4b) Yes, existence of an Euler path (or a Hamiltonian one, etc), such as *bdecbae*, proves it.

I also accepted "Yes, there is a path from any vertex to any other vertex", though this is just a rephrasing of the definition, not really a proof. I gave about 4 points for "Yes, you can get to any vertex" which is too vague (but probably this student had the right idea). Only about 3 points for "Yes, because every vertex has degree at least two". This does not imply connected. The 3 points is mainly for the "Yes". Similar results for "Yes, because every vertex is connected to another vertex". This has the extra disadvantage of using poor vocabulary, creating ambiguity. This *might* be the right idea, expressed very poorly, but I guess this student actually meant "is adjacent to".

4c) A typical good answer was "Yes, see my drawing in which no edges cross".

4d) A typical good answer was "Yes, see my drawing labeled with Blue and Red". For example, you can label vertices a, c and d Blue and the others Red.

4e) No, W_4 has more edges. You could explain using other invariants; the degree sequence, $K_3 \subset W_4$, etc.

5) Yes. $5 + 9 - 2 = 12$, so $12 \in S$. And $5 + 12 - 2 = 15$, so $15 \in S$.

6) There are many good answers. See the text for similar easy examples. You should aim for a "standard" tree here (root at the top, full binary, clearly labeled edges and leaves, etc). I generally gave at least 4 points if your T clearly produced an acceptable code. If there was no discernible method, the grading was harder, even with a good code - one reason for using a standard tree.

7a) Yes. If strings x and y agree in the first three digits, and so do y and z, then so do x and z.

7b) No. For example, 11100 R 11101 and 11101 R 11100, but $11100 \neq 11101$. It is *not* enough to say that R is symmetric. For most people, explaining this in words seemed difficult, but I accepted "x and y might agree in the first three digits, but disagree in the last two" (even better with the definition included).

7c) Each class contains 4 bit strings (the last two digits can vary).

8) $2^{(2^3)} = 2^8$. The 8 is the number of rows in the "truth table". Of course, $(2^2)^3 = 4^3$ is not the same.

9) The tree should have $+$ as a root, with children 3 and $\div$, etc. The RP (or postfix) notation is $3yx \div +$.

10) Combine 111 with 101 to get 1-1 = xz, etc. Get $F = xy + xz + \bar{x}\bar{y}\bar{z}$.

Bo) Use I.E. and the standard eggs-into-baskets formula (three times) to get $|A_0 \cup D_1| = |A_0| + |D_1| - |A_0 \cap D_1| = C(5,3) + C(4,2) - C(3,2) = 10 + 6 - 3 = 13$.

You might also be able to solve this smallish problem by listing, though that was not the intention. For example, $A_0 \cap D_1$ is 0,2,0,1 and 0,1,1,1 and 0,0,2,1 so $|A_0 \cap D_1| = 3$.