Sample Problems from Discrete Math exams
S Hudson, 2008 and 2015

This page was actually written as a review of Discrete Math for students starting my senior level Combinatorics class in 2008. So, I omitted some easy problems, but included bonus questions. These are slightly harder on average than ones on my old finals.

You are expected to understand induction/recursion, divisibility and modular arithmetic, sets and functions (in the abstract) and basic principles of counting. I've collected problem from approx 3 finals but deleted problems about graphs, logic, relations, etc. So, the numbering below may look odd. I don't have a complete answer key, but have inserted answers to the first few problems.

1) In how many ways can 3 men and 3 women be seated;

a) in a row with no seating restrictions

Answer: 6!=120. I usually don't insist on simplifying - you can leave the answer as 6!. Of course, people are considered distinct, so there are 6 choices for the first person, 5 for the next, etc. These numbers multiply out to 6!. If the problem were about 3 red balls and 3 blue balls instead, the answer would be different; C(6,3)=20.

b) if men and women must alternate in a row

Answer: 2x6x6=72. Decide whether to start with a man or woman (2 options), then decide how to order the men (3! = 6 options), and then the women (6 again). If you can organize a problem as a sequence of decisions, it is usually easy.

c) if Mr. and Mrs. Smith (2 of the 6 people) must sit together

d) if Mr. and Mrs. Smith may not sit together

e) in a circle where no two women sit together.

2) How many strings are there of four lowercase letters that have the letter x in them?

Answer: $26^4 - 25^4$. An example of such a string (an ordered list of 4 letters) is $txrx$. We aren't told how many x's to include, etc, so this problem seems hard to organize. But it is easy to count how many do NOT have an x in them, and subtract.

3) Answer with True or False.

a) For any matrix A, the matrix A^tA is symmetric.

b) Every nonempty set of integers has a least element.

c) $n!$ is $O(n^n)$.

d) Any postage greater than 4 cents can be formed using just 3-cent and 5-cent stamps.

e) If $\Sigma = \{a, b, c, d\}$, then Σ^* is countable.

Answer: TFTFT

4) How many numbers in the set $\{1, 2, 3, \ldots 10000\}$ have the property that the sum of their digits is 7? (for example 52: 5+2=7). It may help to consider the related problem of placing 7 eggs into a certain number of baskets.

6) Show that $3x^3 + 2x^2 + x + 1$ is $O(x^3)$.

7) Prove that $n^5 - n$ is divisible by 10 for all positive integers n. Hint: Show it is divisible by 2 and by 5 separately. One of these is fairly easy (consider odd and even numbers). Use induction on the other one. You should encounter the term $(n + 1)^5$ along the way, which you should multiply out, perhaps using the binomial theorem.

Answer: Basic divisibility theory tells us the Hint is a good plan (if this is not common sense to you, then read up on this subject). Also, it is clear that $n^5 - n$ is even in both cases (n is odd, or n is even). So, the main claim is that $n^5 - n$ is divisible by 5, for all positive integers n. As you probably know, a claim like this one, with the phrase *for all positive integers n*, can often be proved by induction. There are two steps:

Basis Step: Let $n = 1$ (the smallest option). Then $n^5 - n = 0$, which is divisible by 5 $(0 = 0 \cdot 5)$, so the claim is true in this case.

Induction Step: Let $n \geq 1$ and assume the claim is true for n, so $5|n^5 - n$ (this is the induction hypothesis). We must prove it is true for $n+1$. We must prove $5|(n+1)^5-(n+1)$. From the Binomial Theorem (you should know this!),
$(n+1)^5-(n+1) = n^5+5n^4+10n^3+10n^2+5n+1-n-1 = [n^5-n]+[5n^4+10n^3+10n^2+5n]$
We know 5 divides the first group on the right (by the induction hypothesis) and it clearly divides the second group, too. So, it divides their sum. We have proven the induction step and are done.

Extra Credit;

A) (5 pts) Mr. and Mrs. Smith invite 5 other couples over for dinner. Upon arrival, some people shake hands. No one shakes hands with themself or their spouse. Mr. Smith asks each of the 11 other people how many people they shook hands with. He gets 11 different answers. How many people did Mrs. Smith shake hands with? Justify your answer.

Partial Answer: As I recall, he shakes hands with one person from each other couple (5 hands), but it's tricky to prove that. Show that the two people who answered 0 and 10 must be a couple, and 'remove them', and repeat.

B) (5 pts) A dominoe is a pair of identical squares. Each square is blank or has a number 1,2,3,4,5, or 6. How many different dominoes are there? [Note that a dominoe is a "non-ordered pair", so (2,5) is the same dominoe as (5,2)].

2) (10pts) A bowl contains 5 red balls and 7 silver ones. A woman chooses balls at random without looking at them. How many must she choose, to be sure of having at least 3 of the same color? Briefly justify your answer using the Pigeonhole principle.

3) (10pts) How many functions f are there from the set $\{a, b, c\}$ to the set $\{1, 2, 3, \ldots n\}$ such that $f(a) = f(b)$?

5) (15pts) Give an induction proof that $f_1 + f_3 + \ldots + f_{2n-1} = f_{2n}$ when n is a positive integer (where $\{f_n\}$ is the Fibonacci sequence, $f_0 = 0$, $f_1 = 1$, etc).

6) (10 pts) Give a recursive definition of $n!$ for $n \geq 0$.

7a) (10 pts) In Problem 1, A is an example of a 3x3 zero-one matrix. How many 3x3 zero-one matrices are there?

7b) (5 pts) How many of those are symmetric?

8) (15 pts) Use mathematical induction to prove:

$n^3 - n$ is divisible by 6 whenever n is a positive integer. (you can do div. by 3 and by 2 separately).

Answer/remark: This is very similar to a problem answered above. Try it yourself!

1) (15pts) a) How many functions are there from a set of 3 elements to a set of 8 elements?

Answer: 8^3. This is an easy 'sequence of decisions' problem [Ask $f(a) = ?$, $f(c) = ?$, etc].

b) How many one-to-one functions are there from a set of 3 elements to a set of 8 elements?

c) How many onto functions are there from a set of 3 elements to a set of 8 elements?

3) (15pts) Let f_n be the n-th Fibonacci number ($f_0 = 0$, $f_1 = 1$, $f_2 = 1$ etc.). Use induction to show that $f_1 + f_3 + \ldots f_{2n-1} = f_{2n}$ whenever n is a positive integer.

5) (10pts) How many bitstrings of length 6 have more zeroes than ones?