HW 1.25

Ch.1.25: Given $G, 2 \leq t \leq p-1, k$ such that if $H \subset G$ with $|V(H)| = t$ then $|E(H)| = k$. Prove that G is complete or trivial.

Plan: We can change H slightly by swapping out two vertices at a time, without changing the number of edges. This should give a contradiction unless G is one of the extreme cases above (but this does seem to take some work - if you see an easier approach please share!).

Proof: We can assume $0 < k < C(t,2)$ to get a contradiction. Choose any H as above, and vertices $w \in H$, $u \notin H$. First, suppose w is not adjacent to u. Let $\hat{H} = H$, except that it includes u but not w. We have not changed the number of vertices, so $|E(\hat{H})| = |E(H)| = k$. So, $|N(w) \cap H| = |N(u) \cap H| =$ some constant C. Secondly, similar reasoning shows $|N(w) \cap H| = C - 1$ iff w is adjacent to u. Deduce that $wu \in E(G)$ for all $w \in A = \{w \in H : |N(w) \cap H| = C - 1\}$ and all $u \notin H$ (or, write $u \in \bar{H}$, a slight abuse of notation). Since this is also true in $\hat{H}$ (formed by swapping 2 vertices from A and $\bar{H}$), deduce that $xy \in E(G)$ whenever $x, y \in A \cup \bar{H}$. Likewise, $xy \notin E(G)$ whenever $x \in H - A$ and $y \in \bar{H}$ (or $y \in A$).

Similar comments hold when $\hat{H}$ is formed from swapping $x \in H - A$ with $y \in \bar{H}$. Since x is now in $\bar{H}$, x must be adjacent to the other vertices in $\bar{H}$ (and in A). But this contradicts the end of the previous paragraph.

I leave a little to you, for example, can $\bar{H}$, A or $H - A$ be empty?