MAD 3305 May 22, 2014
Exam I and Key Prof. S. Hudson

You should also have a separate sheet of paper showing some graphs you will need. [I may post this separately on my exam page if time permits] The exam should take about one hour, to be followed by a lecture. Most of the problems are worth 15 points each.

1. Apply the Graphical Sequence Alg to show that the sequence 4, 4, 3, 3, 2, 2 is graphical, and find a graph G that fits it. Your work should be well-organized, to show each step clearly. This will affect your grade as much as your final graph G.

2. Prove: If G is a nontrivial graph, then it has two vertices which are not cut-vertices.

3. Answer each with True or False:

The breadth first search algorithm (BFSA) is $O(|V|)$.

If G has no cycles and $|E| = |V| - 1$, then G is a tree.

Every nontrivial tree has at least two leaves.

K_3 is an induced subgraph of $K_{3,3}$.

If G is a regular bipartite graph then $|V|$ must be even.

4. Choose ONE proof (roughly the one given in class, or see me):

4a. $k(G) \leq k_1(G)$

4b. State and prove the main theorem that justifies Prim's Alg.

4c. A connected graph G is a tree iff $|E| = |V| - 1$.

5. Apply Dijkstra's Alg clearly to Fig.1 (see other sheet) to find the distance and shortest path from x to y.

6a. See Fig.2. Is it weakly connected ? unilateral ? strongly connected ?

6b. Prove your answer to the last question, on whether it is strongly connected.

7. You are given a 5-gallon jug, a 2-gallon jug and a faucet but no other measuring tools. Find 2 distinct ways (no repeating, etc) to get exactly 4 gallons in the larger jug. For full credit, use a graph to solve this.

Bonus (about 5 points): Give an example of a non-trivial tree T such that $\overline{T}$ is also a tree.

Remarks and Answers: The average was 70 out of 100, based on the top 12 of 14 scores. The highest scores were 88 and 80. The results on all the problems were fairly good except for Problem 2 (only 18% correct, including partial credit). The (advisory) scale is

A's = 78 to 100
B's = 68 to 77
C's = 58 to 67
D's = 48 to 57

1) The GSA reduces the sequence to 0,0, which is graphical. It is fairly easy to get G by working backwards from that.

2) Very few people found a reasonable plan for this proof (though it was assigned as HW, Ch. 2.28). Here is one plan; the details / justifications are left to you.

Outline: We can assume G is connected. Choose a vertex x and let y be a vertex as far from x as possible. Then it is not a cut vertex (why not?). Now replace x by y to find a second one. [Other reasonable plans are to use the leaves of a spanning tree $T \subset G$, or to use induction on $|V|$.]

3) FTTFT

4) Nobody chose 4b, with a roughly equal split 4a vs 4c. See the text or lectures for the proofs. For 4a, our proof reserves e_1 as special, and I did not see any completely successful proofs without that. For 4c, there are two parts; several people proved "only if" but forgot to do "if".

5) $d(x, y) = 11$ and the path is x,b,d,j,y. For full credit you had to show each step of Dijkstra's algorithm fairly clearly. That should include all the distances from x, such as $d(x, h) = 7$, and a steadily growing set of understood vertices, etc.

6a) Yes.Yes.Yes. Several people wrote only that it is strongly connected, but did not explicitly answer the first two questions. If you know that this implies the others, you should say so!

6b) It is SC because it has a closed spanning walk. For example, abdefca.

7) [10 points] Most people created a diagram similar to Fig 2.4.3 in the text, and got full credit. I feel that a couple of sentences of explanation are in order here, but I did not deduct points for that this time. Here are the vertices of the 2 paths:

Path One: (0,0) (0,2) (2,0) (2,2) (4,0)

Path Two: (0,0) (5,0) (3,2) (3,0) (1,2) (1,0) (0,1) (5,1) (4,2)

Note that Path Two should not end, for example, with (0,2) (2,0) (2,2) (4,0) because that essentially repeats the first solution.

Bonus) A "Z" shaped subgraph of K_4 works (eg $T = P_4$), and this is the only answer, up to isomorphism.