MAD 3305 June 20, 2014
Final Exam Prof. S. Hudson

The exam should take about 2.25 hours. Most problems are 10 points each, unless labeled.

1) [5 pts] Give an example of $G$ such that $k(G) < k_1(G)$, if possible. If not, explain.

2a) State the definition of the *center* of a connected graph, $Z(G)$.

2b) Choose ONE:

Show that in a tree $T$, the center $Z(T)$ contains at most 2 vertices.

Show that in any connected graph, $Z(G)$ is contained in a single block of $G$.

3) Show that every planar graph with $p \geq 4$ vertices contains at least 4 vertices with degree at most 5.

4) [5 pts] Draw a planar embedding of $G = K_{2,3}$. Then, draw its dual, $G^*$.

5) State Konig's Thm about covers. Use it to prove Hall's Thm about matchings.

6) [5 pts] Draw a 3-regular graph with a cut-vertex and a perfect matching (and identify the vertex and the matching).

7) [5 pts] Give an example of $G$ such that $\alpha(G) \neq \beta_1(G)$; if not possible, then explain.

8) [5 pts] Specify (*list* them, roughly speaking) which graphs are 3-critical, and justify your answer.

9) Use Birkhoff's thm to find the chromatic polynomial of $G = K_5 - e$.

10) [8 pts] Explain briefly what the *genus* of a graph is. Explain briefly why $\gamma(K_{3,3}) = 1$. For max credit, draw an embedding, not necessarily planar. State the revised Euler formula for non-planar graphs.

11) True-False. You can assume the graphs below have at least one edge.

If $G$ contains no odd cycles, then $\chi(G) = 2$.

The Petersen graph is neither planar nor Hamiltonian.

None of the five regular polyhedra are bipartite.

If $G$ is bipartite, then it is class 1.

In any connected graph $G$, $p \leq q - 1$.

12) Choose ONE proof:

a) State and prove Menger's Thm #2 (about edge-disjoint paths).

b) Prove this part of Berge's Thm; if there is no $M-$augmenting path in $G$, then $M$ is a maximum matching. You may use the Lemma about the components of $H$ without proving it (but do state it).

c) Prove that if $\delta(G) = 1$ and $\Delta(G) = 2$ and the graph is connected, then it is (isomorphic to) a path.

13) [7 pts] Use the stable matching algorithm (with women proposing) to find a stable matching. The first matrix $W$ shows the preferences of the women $A, B, C, D$, [the rows] for the men $a, b, c, d$, [the columns]. The second matrix shows the preferences of the men. For example, woman $D$ ranks man $d$ as second best, while $a$ ranks $D$ as third best. Show all your work in some organized manner.

$$W = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 1 & 4 & 3 & 2 \end{pmatrix} \qquad M = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 4 & 2 & 1 & 2 \\ 1 & 3 & 3 & 4 \\ 3 & 4 & 4 & 3 \end{pmatrix}$$

**Remarks and Answers:** The average was about 55%, slightly better than on Exam 2, but with a high score of 90. The best results were on the Give-an-example type problems and the worst were on the proofs, both the HW kind and the textbook kind. The scale for the semester turned out pretty close to the second one on the Exam 2 key.

1) Glue two cycles together to form (roughly) a figure 8 with a cut vertex. So $k = 1$ but $k_1 = 2$.

2a) $v \in Z$ iff ecc$(v)$ = rad $(G)$. [that is, ecc$(v)$ is minimal.]

2b) Option 1: ETS that any two vertices in $Z$ are adjacent (so, if $|Z| > 2$ then $T$ would contain $K_3$). Suppose not, that $z_1, z_2 \in Z$ but the unique $z_1 - z_2$ path in $T$ contains a 3rd vertex, $z_3$. Argue that ecc $(z_3)$ is smaller than rad $(G)$ (details left for you), a contradiction. Option 2 was done in class.

3) This was assigned HW, and is answered on Prof Ram's site. I don't know of any simple induction proof (though it was a popular try, and the idea does seem plausible).

Pf: WLOG $G$ is max planar, so that $q = 3p - 6$, and $\delta(G) \geq 3$. Assume $G$ does *not* have 4 vertices of small degree as claimed, so that the degree sequence ends with 6, 3, 3, 3 (or larger numbers). By our first thm, $2q = \sum \deg(v) \geq 6(p-3)+9 = 6p-9$. This contradicts $q = 3p - 6$.

4a) Draw a square, then a diagonal edge, and then insert a 5th vertex on the midpoint of that edge. There are other pictures, but yours should have 3 regions.

4b) It looks like $K_3$ except that every edge is drawn twice (a multigraph).

5) If bipartite, $\alpha = \beta_1$. This implies some $|C|$ = some $|M|$.

Pf: Assume this and 'neighborly' (see lectures). ETS: $|M| = |X|$. Since $|C| = |M| \leq |X|$, ETS: $|C| \geq |X|$. From definition of a cover, $N(X \setminus C) \subseteq C \cap Y$. From 'neighborly',

$|C \cap Y| \geq |N(X \setminus C)| \geq |X \setminus C|$. So, $|C| = |C \cap X| + |C \cap Y| \geq |C \cap X| + |X \setminus C| = |X|$. Done.

6) Already answered on a HW web page.

7) This is not possible if $G$ is bipartite, but let $G = K_3$, $\alpha = 2$, $\beta_1 = 1$ (and there are many other examples).

8) Odd cycles such as $C_5$. It is clear (we discussed this) that they are 3-critical. Conversely, suppose $G$ is 3-critical. So, it is not bipartite, and contains a (minimal) odd cycle $C$. If there were a $v \in G - C$ it would be removable (contradiction to 3-critical). If $G$ contained an edge $e \notin C$, with vertices in $C$, then it would contain a cycle smaller than $C$ (contradiction). So $G = C$.

9) $P = P_{K_5} + P_{K_4} = k(k-1)(k-2)(k-3)[(k-4)+1]$ and simplify a bit.

10) See lecture notes (very briefly, $K_{3,3}$ embeds in the torus but not the plane).

11) TTFTF

12) In my opinion, (a) is easiest, as done in class. The most popular choice was (c), though it was new. The results were not very good. People tended to ramble, with fairly plausible comments, but without a definite proof strategy. One decent idea is to use induction on the number of edges, though I didn't see this done very cleanly. The shortest cleanest proof I can find is:

Pf: Let $P$ be a path of maximal length in $G$. ETS $G = P$. If $G$ contains some vertex $v \notin P$ then there is a shortest $v - w$ path $P_2$ with $w \in P$. We get a contradiction in both cases; if $w$ is a leaf of $P$ then $P \cup P_2$ forms a longer path than $P$. If it isn't, then deg $(w) > 2$ in $G$. So there is no such $v$. If there is an edge $e$ in $G - P$, then $e$ must connect the two leaves of $P$. So $G$ is a cycle, which contradicts $\delta(G) = 1$.

13) It takes approx 5 steps, and ends with Ab, Bc, Ca, Dd. This example is from Brualdi's book on Combinatorics, and it also appeared in class..