MAC 2311 — 4pm, Jan 30, 2009

Exam I — Prof. S. Hudson

1) [30 points] Compute each limit:

a) $\lim_{x\to 2} \sqrt{\frac{2x^2-8}{x-2}} =$

b) $\lim_{x\to +\infty} \tan^{-1}(x) =$

c) $\lim_{x\to +\infty} \sqrt{x^2-1} - x =$

2) [20pt] Short answer problems:

a) Find all values of θ (in radians) that satisfy the equation $\cos(\theta) = \frac{-1}{\sqrt{2}}$

b) Solve $x^2 - 4x + 3 < 0$.

3) (20pts) Answer True or False. You do not have to explain.

a) $\tan(\pi/2 + x)$ is continuous at every point in the interval $(0, \pi)$.

b) If $\lim_{x\to 2} f(x)$ exists, then $\lim_{x\to 2^+} f(x)$ exists.

c) If x is a number so that $|x-2| < 1$, then $|x-4| < 5$.

d) $\forall \epsilon > 0, \exists \delta > 0, \delta + \epsilon < .001$.

e) If $f(1) = 1$ and $f(3) = 3$, and f is continuous, then $f(x) = 2$ for some x in $(1, 3)$.

f) For $f(x) = x^2 + 1$, the slope of every secant line is positive.

g) The domain of $f + g$ is the same as the domain of fg.

h) The derivative of $\tan^{-1}(x)$ at $x = 1$ is positive.

i) If $f(x)$ is continuous, then $g(x) = f(x)/(1 + x^2)$ is also continuous.

j) $f(x) = \sin(x)/x$ has a removable discontinuity at $x = 0$.

4) (10pts) Find the equation of the tangent line to $f(x) = x^2$ at (-1,1).

5) (5pts) Eliminate the parameter. $x = \cos^2(t)$ and $y = \sin^2(t)$

6) [15pts] Choose ONE of the problems below to do. Remember to use enough words and sentences - not just formulas.

a) Prove that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

b) Show that every polynomial of odd degree must have a root.

c) [5pts bonus, if correct] Show that $\lim_{x\to 3} x^2 = 9$ using the definition of limit.

Remarks and Answers: The average was approx 55/100, which is fairly low. The worst results were on the limits on page 1, and the proof (#6). The new [unofficial] scale is:

A's = 75 to 100
B's = 65 to 74
C's = 55 to 64
D's = 45 to 54
F's = 00 to 44

1a) $2\sqrt{2}$, or you can leave it as $\sqrt{8}$. Each part of problem 1 was worth 10 points. For full credit, your method had to be fairly clear, and valid.

1b) $\pi/2$. I accepted a graph as a valid method, since that's how we did this one in class.

1c) 0. Use the conjugate, as usual. If I have time, I will post a graph of $y = \sqrt{x^2 - 1}$ and of $y = x$ on the exam page. It shows that $y = \sqrt{x^2 - 1}$ has $y = x$ as an oblique asymptote (which is consistent with the answer to this problem).

2a) Either $\theta = 3\pi/4 \pm 2\pi n$ or $\theta = 5\pi/4 \pm 2\pi n$, where $n = 0, 1, 2 \ldots$.

2b) $1 < x < 3$. Factor the polynomial and then decide which interval(s) belong in the answer [this is precalc, but see the similar problem from my AM exam for more practice, and few more comments on the method].

3) TTTFT FTTTT. Most of these have simple explanations - see me if any are not clear.

4) $y - 1 = -2(x + 1)$, or simplify that to $y = -2x - 1$. First, compute $m_{\tan} = \lim_{x\to -1} \frac{x^2-1}{x+1} = -2$, and plug that into the standard algebra formula for a line.

5) $x + y = 1$. I made this into a True-False question on the AM exam.

6) See the text or lectures notes. For b) show that the poly has both positive and negative y values using limits at $\pm\infty$ and then quote the IVT. A student asked about c), so I am giving a fairly detailed proof here (also done in class):

Proof that $\lim_{x\to 3} x^2 = 9$. We must show that $\forall \epsilon > 0, \exists \delta > 0$ such that if $0 < |x - 3| < \delta$, then $|x^2 - 9| < \epsilon$.

Let $\epsilon > 0$ be some given number. Set $\delta = \text{Min}[1, \epsilon/1000]$. [I went through the algebra behind this choice in class. Here, I will justify the choice, instead of showing that algebra again, but these two styles are almost the same thing anyway]. Now, we can assume that $|x - 3| < \delta$. Since $\delta \leq 1$ AND $\delta \leq \epsilon/1000$, we can draw TWO conclusions:

a) That $|x - 3| < 1$ and

b) That $|x - 3| < \epsilon/1000$

From a) we get $|x + 3| < 7$ [Please check the algebra for practice]. Combining this with b) we get

$$|x^2 - 9| = |x - 3|\ |x + 3| < (7\epsilon/1000) < \epsilon$$

If you compare this with your lecture notes, you'll see that the steps there are very similar, but almost in reverse order - so that the formula $\delta = \text{Min}[1, \epsilon/1000]$ appears at the end. That is the style favored in our textbook (but I think it is harder to explain properly or to read). Of course, there are other possible proofs, involving other choices for δ.