MAC 2311 November 15, 2007
Exam III and Key Prof. S. Hudson

1) (15 pts) Analyze and graph $f(x) = x^4 + 2x^3 + 2$. Hints: $f(1.5) = 13.81$ and $f(-1.5) = 0.31$. And, remember to -

Show all your work (I also suggest checking it carefully).
Describe the intervals of inc/dec and concavity.
Plot and *clearly label* the critical points, inflection points and asymptotes (if any).

2) (15 pts) Mark each sentence True or False;

$f(x) = \ln(x) - x$ has an absolute maximum on $(0, \infty)$.

If a polynomial has three different roots, then it also has three different critical points.

If c is a critical point of f, then f has a relative extrema at c.

The graph of a rational function can cross a horizontal asymptote.

The function $f(x) = x^2/(x+1)$ has an oblique asymptote.

3) (25 pts) Compute y':

a) $y = e^{\sin(x)}$

b) $y = \ln(\ln(x))$

c) $y = \sec^{-1}(7x)$

d) $\sin(x^3y^2) = x$

e) $y = x^{2x}$

4) (15 pts) Compute each limit:

a) $\lim_{x\to 1} \frac{\ln(x)}{x^3-1} =$

b) $\lim_{x\to 0}(e^x + x)^{1/x} =$

c) $\lim_{x\to 0} \frac{1-\cos(x)}{\cos(2x)}$

5) (10 pts) Prove that $\sin(x) \leq x$ for all $x \in [0, 2\pi]$. Hint: as in your HW, find the max of $\sin(x) - x$. Remember to explain your reasoning.

6) (10pts) A rectangular plot of land will be fenced in with two kinds of fencing. Two opposite sides will cost \$3 per foot. The other two sides will cost \$2 per foot. What are the dimensions of the rectangle with maximum area that can be fenced in at a cost of \$6000 ?

7) (10 pts) Find the relative extrema, and classify them using the specified Test.

a) $f(x) = \sin(2x)$ on $(0, \pi)$, using the Second Derivative Test.

b) $f(x) = (x-3)e^{-x}$, using the First Derivative Test.

BONUS (5 pts): Give an example of a well-known function with two different horizontal asymptotes.

Remarks and Answers: The average was about 65, so you can use the scale on my syllabus, adjusted downwards about 2 or 3 points (A's = 79-100, etc). The semester average, not including HW yet, is about 67, so you can use the scale on the syllabus for that. I've done this calculation for you, on the upper right corner of page one of your exam, but you should check it.

The grades were good on the $f'(x)$ calculations and the graphing problem. They were OK (better than usual) on the limit calculations. They were bad on problems 5 and 7, maybe because the answers required explicit justification, but they were straight out of the book. Prepare for more like these on the final. Also, the TF results were below average.

1) I've posted a graph of f (in blue). My exam page has a link to that.
It decreases on $(-\infty, -1.5]$.
It increases on $[-1.5, \infty)$.
It is concave up on $(-\infty, -1]$ and on $[0, \infty)$.
It is concave down on $[-1, 0]$.
It has critical points at $(-1.5, 0.31)$ and at $(0, 2)$.
It has an inflection point at $(-1, 1)$ and at $(0, 2)$. No asymptotes.

2) TFFTT The grades were unusually low this time, averaging only about 55%, so here are some explanations. I included some clarifying examples below, which are not really necessary. Be warned - they're just 'out of my head' - not carefully checked. Practice with TF before every test, and read over all the theorems and definitions with this kind of problem in mind. This is an excellent study habit, to complement your usual practice with calculation-type exercises.
a) Check the limits at the endpoints of the domain. Let $x \to 0+$ and $x \to \infty$. Both limits are $-\infty$. So, it must have an abs max (and it's at $x = 1$).
b) It must have 2 stat. pts, by Rolle's Thm, but might not have 3. Probably $f(x) = x^3 - x$ is a good counterexample.
c) Standard example; $f(x) = x^3$ at $x = 0$. (See also the answer to problem 1).
d) There is no reason why it shouldn't. Probably $f(x) = 1/x + 1/(x-1)$ is a good example. It crosses the x axis at $x = 1/2$.
e) It must have one, since deg P = deg $Q + 1$. I think the equation is $y = x - 1$.

3a) $\cos x \; e^{\sin x}$ Use the Chain Rule for 3a, 3b, 3c.

3b) $1/(x \ln x)$

3c) $\frac{7}{|7x|\sqrt{49x^2-1}}$

3d) $[\sec(x^3y^2) - 3x^2y^2]/(2x^3y)$ Use Imp. Diff.

3e) $2x^{2x}[\ln x + 1]$ Use Log. Diff.

4a) $1/3$

4b) e^2

4c) $0/1 = 0$ (do not use LHR)

5) This is the assigned HW exercise, 5.4.43. The answer is mainly the calculation suggested in the hint, but also explain the overall idea.

Phase I (calculations): Find the max of $f(x) = \sin x - x$ (as the hint suggests). Set $f'(x) = \cos x - 1 = 0$, and get $x = 0$ or $x = 2\pi$ (not really critical points, since they are endpoints). By plugging these in, we see that the maximum value is $y = f(0) = 0$. (It is not really important that $x = 0$, just that $y = 0$ is the max).

Phase II (explanation): This means $f(x) \leq 0$ whenever $x \in [0, 2\pi]$. So, $\sin(x) - x \leq 0$. So, $\sin(x) \leq x$. Done.

Many people omitted Phase II. I assume they didn't quite understand it, but it is just as important as Phase I. Some of the proofs were very vague - if yours seems to prove $\sin(x) \leq x$ for *all* x it must be wrong, since $\sin(-\pi) = 0 > -\pi$, for example.

6) Let x be the width of the rectangle, in feet, at \$3 per foot, and let y be the height (other choices are OK; just show your choice clearly). So, $6x + 4y = 6000$ and $A = xy = x(6000 - 6x)/4$. From $A'(x) = 0$, get $x = 500$ and $y = 750$.

7) These are problems 29 and 30 from Ch 5.2. You are supposed to answer these without drawing the graphs, but that's a good way to check your answers. I've posted the graphs for you (the link is on my exam page). 7a is in red; 7b is in green.

a) $x = \pi/4$ is a rel max, because $f''(\pi/4) < 0$. Also, $x = 3\pi/4$ is a rel min, because $f''(3\pi/4) > 0$.

b) $x = 4$ is a rel max, because f' changes from + to -.

Bonus) I expected $\tan^{-1} x$, which I've drawn on the board a few times, but other answers are acceptable (such as $|x|/x$).