MAC 2311 April 24, 2007

Final Exam Key Prof. S. Hudson.

1) (20 pts) Compute and simplify;

$\int 2^x \; dx =$

$\int \frac{1}{1+x^2} \; dx =$

$\int \tan(x) \; dx =$

$\int \cos^2(x) \; dx =$

2) (30 pts) Compute;

$\lim_{x\to\infty}(1+2/x)^x$

$\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$

Find $dy/dx$, given that $y = x^2 \ln(3x+2)$

Find $dy/dx$, given that $y = \sin^{-1}(1/x)$

Find $dy/dx$, given that $x^2 + y^2 = 100$

Find $dy/dx$, given that $x(t) = t^2 + 1$ and $y(t) = \sin(t)$.

3) (10 pts) A firm decides that it can sell $x$ units of its product daily at a price of $p$ dollars per unit, where $x = 1000 - p$. The cost of producing $x$ units per day is $C(x) = 3000 + 20x$.

3a) Find the profit function $P(x)$.

3b) Assuming a production capacity of at most 500 units per day, how many units must the company produce per day to maximize profits?

4) (20 pts) Answer TRUE or FALSE:

If F is an antiderivative of f on (2,3), then $F$ is continuous on (2,3).

If $f'(x) > 0$ and $g'(x) > 0$ (on the real line) then $(fg)'(x) > 0$ too.

If $f$ and $g$ are both increasing (on the real line) then $fg$ is also increasing.

If f is a polynomial, then it is either increasing or decreasing on the interval (0,1).

If F and G are two antiderivatives of f then they differ by a constant.

For all $x \in (-1, 1)$, $\sin^{-1}(x) + \cos^{-1}(x) = \pi/2$.

The function $\csc(x)$ is continuous on the interval $(-\pi/4, \pi/4)$.

If f is differentiable and f(2)+3=f(3)+2 then there is a point $c \in (2,3)$ where $f'(c) = 0$.

If $f'(3) = 0$ then $x = 3$ is a critical point of $f$.

The maximum value of $f(x) = |x - 3|$ on [0,10] exists, and it occurs at a critical point of $f$.

5) (10 pts) Find and classify the relative extrema of $f(x) = \sin(2x)$ on $0 < x < \pi$, using both the first and second derivative tests.

6) (10 pts) CHOOSE ONE;

A) State and prove Rolle's theorem.

B) State and prove the Product Rule.

BONUS: Suppose that $x(t) = t\sin(t)$ and $y(t) = t\cos(t)$. Compute the *second* derivative $d^2y/dx^2$ when $t = \pi/2$.

**Remarks and Answers:** The results were not good. The average was about 54/100, based on the dozen or so students with grades over 40 (but not including the highest grade). Even those students scored under 50 per cent on problems 1 (antiderivatives) and 5 (1st/2nd Deriv tests). The True False results were under 60 per cent. I did not choose a scale for the exam, but for the semester grades, I used: A's = 75 to 100, B's = 65 to 74, C's = 55 to 64, D's = 45 to 54, F's = 00 to 44.

1) Antiderivatives; avg 9.3/20

$2^x / \ln 2 + C$
$\tan^{-1} x + C$
$\ln|\sec x| + C$
$x/2 + (\sin(2x))/4 + C$.

2) Limits and Derivatives; avg 19/30

$e^2$
$1/2$
$y' = 2x\ln(3x+2) + 3x^2/(3x+2)$
$y' = -1/(x\sqrt{x^2-1})$
$y' = -x/y$
$y'(t)/x'(t) = \cos(t)/(2t)$

3) Max-min economics; avg 5.3/10.

Recall that revenue is $R(x)$ = price times sales = $p \cdot x$. You should substitute $p = 1000 - x$, since I asked you to answer in terms of $x$ (not the other way!). Also, be careful to put parentheses around the formula for $C(x)$. This should not be a hard problem, but there were a lot of silly mistakes.

a) $P(x) = R(x) - C(x) = (1000 - x)x - (3000 + 2x) = -x^2 + 980x - 3000$.

b) The candidates are $x = 0,\ 490,\ 500$, and the max occurs at $x = 490$.

4) Average 11.7/20, TFFFT TFFTF (you can ask about these)

5) Using 1st+2nd D.Tests, avg 3.3/10.

Set $0 = f'(x) = 2\cos(2x)$ and get $x = \pi/4$, or $x = 3\pi/4$ (few people found this second critical point). The $f'$ line goes +,−,+, so the first critical point is a rel max and the second is a rel min (1st DT). And $f''(x) = -4\sin(2x)$ so $f''(\pi/4) < 0$, so this is a rel max. And $f''(3\pi/4) > 0$, so this is a rel min (2nd DT). You should not draw a $f''$ number line (it doesn't hurt, but it doesn't directly answer the question).

6) Proof, avg 5.3/10. See the textbook. As usual, you need to understand (and probably rehearse) these proofs before the exam. For Rolle's thm, you need to think carefully about the wording.

The Product Rule proof is a fairly direct calculation, but you should still explain most of the steps ('by definition of derivative', 'by continuity', 'by a theorem about limits', etc). Several people wrote 'by definition of limit' in various places, but that is not used in this proof.

Bonus: 2nd deriv, from parametric eqns, avg 0.4/5.0

We did one of these in class, but maybe this example was a little too messy. I don't think anyone came very close to the answer, so I gave a point or two for a good start (see Ch 11.2). I got $-2 - \pi^2/4$.