MAC 2312 May 27, 2016
Exam I Prof. S. Hudson

1) (5pts each) Compute each sum and simplify, if reasonable (but an answer like 2^{2015} or $\frac{2}{3}+\frac{4}{7}$ or $\ln(3)$ is OK).

$\sum_{k=1}^{20} 3k(k+2)$

$\sum_{k=0}^{20} 2^k$

$\sum_{k=1}^{2016} \ln(\frac{k+1}{k})$

2) [5 pts] Simplify $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} \ln(t)\,dt$, removing all h's and t's, with a brief explanation. Recalling the proof of FTC#2 might help, but there are probably other methods.

3) (5pts each) Compute (by substitution, geometry, FTC, etc):

$\int_0^1 \frac{dt}{\sqrt{1-t^2}}$

$\frac{d}{dx}\int_2^{e^x} |10-t|\,dt$

$\int_0^{\pi/6} \sin^2(x)\,dx$

$\int_0^2 |x^2-1|dx$

$\int_{-1}^1 x\sqrt{\cos(x^2)}dx$

4) (10pts) A rock is dropped from the top of Green Library, at a height of 256 feet. a) Use integration to find a formula for its height in feet, $s(t)$, which applies from time $t=0$ until it hits the ground.

b) How long does it take for the rock to hit ground ? You can assume gravity is -32 ft/sec^2.

5) (10pts) Estimate the area under $y=x^2$ from $x=1$ to $x=3$ using a Riemann sum with $n=4$ and the Right Endpoint Rule.

6) (10pts) Let R be the region bounded by $x=0$, $y=3$ and $y=2x$. Find the volume of the solid generated by revolving R around the y-axis. Use the Disk, Washer or Shell Method and a definite integral (no geometric shortcuts allowed!).

7) (15pts) Answer True or False to each part. You do not have to explain.

For all numbers $b>10$, the average value of $\sin(x)$ on $[0,b]$ is nonnegative.

The function $f(x)=\frac{1}{x-1}$ is integrable on $[0,3]$.

The distance traveled by a particle in rectilinear motion is always nonnegative.

For any two functions $f,\ g$ defined on $[0, 2\pi]$, $(f - g)_{ave} = f_{ave} - g_{ave}$

For any two functions $f,\ g$ defined on $[0, 2\pi]$, $(f \cdot g)_{ave} = f_{ave} \cdot g_{ave}$

8) (10pts) Choose ONE proof. If possible, use sentences or formulas with complete justifications. Recall that the grading will be based on the clarity of your logic and explanations, as much as on any calculations involved.

A. State and prove the theorem about $\ln(ab)$.

B. State and prove the geometric sum theorem.

C. State and prove the Mean Value Theorem. This one is likely to be harder, so I may grade a bit easier, or even give extra credit for very good answers to it.

Bonus: (5 points, maybe hard): Find a function $f(x)$ such that $\int_0^t f(x)\ dx = e^{3t} - 1$ for all x.

Remarks and Answers: The average score among the top half of the class was 70, with highs of 86 and 85. The best results were on the proof (approx 90% correct) and the worst were on problems 1 and 6 (sums and volume, approx 60% on each) though I didn't think those two were the hardest. The advisory scale for the exam is:

A's 77 to 100
B's 67 to 76
C's 57 to 66
D's 47 to 56

1a) $3\sum_{k=1}^{20} k^2 + 6\sum_{k=1}^{20} k = 3 \cdot 20 \cdot 21 \cdot 41/6 + 6 \cdot 20 \cdot 21/2 = 8610 + 1260 = 9870$. Though the simplification seems reasonably easy, I decided not to count it this time.

1b) $2^{21} - 1$. The sum is geometric with $a = 1$, $r = 2$, $n = 20$. I expected this to be easy, but many people did not recognize this as geometric, or did not know the formula for S from the HW (and it was one of the announced proofs for this exam).

1c) $\sum_{k=1}^{2016} [\ln(k + 1) - \ln(k)] = \cdots$ (telescoping) $\cdots = \ln(2017) - \ln(1) = \ln(2017)$.

2) $\ln(x)$. The problem matches the last half of the proof of FTC#2, if $f(x) = \ln(x)$.

The answers were pretty good, considering that this problem is a bit unusual. Common mistakes were writing the final answer as $f(x)$, or not getting to any clear final answer, or not showing enough work (for example, a few people just wrote '$\ln(x)$ by FTC#2').

3a) $F|_a^b = \sin^{-1}(1) - \sin^{-1}(0) = \pi/2$.

3b) $|10 - e^x|e^x$ using the FTC#2 and the Chain Rule.

3c) $\frac{1}{2}\int_0^{\pi/6} 1 - \cos(2x)\ dx = \frac{1}{2}[\frac{\pi}{6} - \sin(\pi/3)/2] = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$.

3d) $\int_0^1 1-x^2\ dx + \int_1^2 x^2-1\ dx = \cdots = 2$. We use the abc thm to remove the abs vals. We use $b=1$ because $x^2-1=0$ when $x=1$.

A graph is not required, but is useful as a rough guide. A few people drew the curved lines as straight lines, used the area formula for triangles, and by luck this gave the right answer (for partial credit). Usually geometry formulas don't help with parabolas / etc.

3e) $\int_{-1}^1 x\sqrt{\cos(x^2)}dx = 0$ because $x\sqrt{\cos(x^2)}$ is odd. Some people also got full credit using $u=x^2$, though I doubt that method works in all similar problems.

4a) $s(t) = 256-16t^2 = 16(16-t^2)$.   4b) t=4.

5) The x_k are 1.5, 2, 2.5 and 3 (starting at $k=1$). The $f(x_k)$ are 2.25, 4, 6.25 and 9. Add these and then multiply by $\Delta x = 0.5$ to get $43/4 = 10.75$.

It is OK to use general formulas such as $x_k = a + k\Delta x$ and Σ notation. Most people did. But that notation is mainly intended for problems with a bigger n and perhaps a limit at the end. For this simpler kind of problem, it seems better to draw a graph with 4 rectangles. Use it to guide your calculations and to check for obvious mistakes.

For example, many people got Δx wrong in the very first step, which ruined the rest of the work. Many forgot to square the x_k.

6) The Disk method with dy seems a little easier than the Shell method with dx, but you cannot use the Washer method. With disks, $V = \int_0^3 \pi(y/2)^2\ dy = \frac{\pi}{4}\frac{27}{3} = \frac{9\pi}{4}$.

The most successful answers included a well-labeled picture of a cone, and a rectangle, and a clear choice of the method to be used. In fact, throughout the exam, people who drew pictures tended to do the best (see my remarks on problem 5, etc).

7) TFTTF.

8) See the text or lecture notes. The most common choice was the $\ln(ab)$ proof, and the answers were mostly good. Most people took care to include and explain each required step.

Bonus: Take a derivative of both sides using FTC#2 to get $f(t) = 3e^{3t}$ (or, $f(x) = 3e^{3x}$ is the same thing). Check this, especially if you aren't sure a correct answer actually exists. For example, if we change the problem to $\int_0^t f(x)\ dx = e^{3t}$, there is no answer (try setting $t=0$).

Approx 10 people got at least partial credit on this, which is pretty good for a bonus problem.