MAC 2312 Feb 2, 2017
Exam I + Key Prof. S. Hudson

1) (5pts each) Compute each sum, using a summation formula if possible. Simplify, within reason (fractions are OK, but do not leave a $\sum$ symbol, or many + signs in your answers).

$\sum_{k=5}^{55} \frac{2}{k} - \frac{2}{k+1}$

$\sum_{k=1}^{20} k(k+2)$

$\sum_{k=0}^{8} \cos(k\pi)$

$\sum_{k=1}^{30} |10-k|$

2) (5pts each) Compute each (by substitution, geometry, FTC, inv. trig, etc):

$\int_0^{\pi/6} \sin^4(x)\cos(x)\ dx$

$\int_{0.5}^{2} \sqrt{2x+1}\ dx$

$\int_0^3 \sqrt{9-t^2}\ dt$

$\frac{d}{dx}\int_x^2 |8-t|\ dt$

$\int_0^1 \frac{1}{1+x^2}\ dx$

3) (10pts) A rock is dropped from the top of Green Library, at a height of 256 feet. a) Use integration to find a formula for its height in feet, $s(t)$, from time $t = 0$ until it hits the ground.

b) How long does it take for the rock to hit ground ? You can assume gravity is -32 ft/sec^2.

4) (10pts) Estimate the area under $y = x^2 + 1$ from $x = 1$ to $x = 3$ using a Riemann sum with $n = 4$ and the Right Endpoint Rule. Do not use any anti-derivatives in this problem (except maybe to check your work).

5) (10pts) Let R be the region bounded by $x = 1$, $x = 2$ $y = 0$ and $y = x^2$. It looks like a trapezoid, but with a curved top. Use the disk method to find the volume of the solid generated by revolving R around the x-axis. You can use standard formulas, and do not have to mention Riemann sums.

6) (15pts) Answer True or False to each part. You do not have to explain.

The function $f(x) = \tan(x)$ is integrable on $[0, 1]$.

The average velocity of a particle is always nonnegative.

If $f(x)$ is continuous, then it has an anti-derivative, $F(x)$.

The function $f(x) = \frac{x-2}{x(x-3)}$ is integrable on $[1, 5]$.

If $f(x) = \sin(kx) - 1$ (for some k) then on $[\pi, 2\pi]$, f_{avg} must be negative.

7) (10pts) Choose ONE proof. If possible, use sentences or formulas with complete justifications. Recall that the grading will be based on the clarity of your logic and explanations, as much as on any calculations involved.

A. State and prove the FTC #2 (about $\frac{d}{dx}\int$).

B. State and prove the formula for $\ln(ab)$.

Bonus: (5 points): How did we define 2^π in class ? If you use anything else we learned that day, such as e^3 or $\ln(4)$, explain those briefly too.

Remarks and Answers: The average among the top half was approx 65 out of 100, which is pretty normal. The two best scores were 99 and 95. The results were similar on all the problems except the bonus, and maybe problem 3 (52%) and problem 5 (78%). Here is an advisory scale for the exam:

A's 74 - 100
B's 64 - 73
C's 54 - 63
D's 44 - 53

1a) Telescoping, $(\frac{2}{5} - \frac{2}{6}) + (\frac{2}{6} - \frac{2}{7}) + \cdots + (\frac{2}{55} - \frac{2}{56}) = \frac{2}{5} - \frac{2}{56}$.

1b) Split it into two easy sums, $\frac{20\cdot 21\cdot 41}{6} + 20 \cdot 21$.

1c) Since $\cos(k\pi) = \pm 1$, this is $1 - 1 + 1 \cdots + 1 = 1$.

It is perhaps debatable whether these answers should be simplified. They are pretty easy, so I'd mildly suggest simplifying, to be on the safe side. I required it on 1c, but not on 1a or 1b.

1d) Like 1a) and 1c) it is best to write out the terms, like $9+8+\cdots+1+0+1+\cdots 19+20$. Then $\sum_{k=1}^{9} k + \sum_{k=1}^{20} k = 45 + 210 = 255$.

2a) $\frac{1}{160}$. If you have not learned that $\sin(\pi/6) = 1/2$ yet, do that now, or you may lose two points again on many more problems. See 1c) and 2e), for example.

2b) $\frac{5^{3/2} - 2^{3/2}}{3}$ from $u = 2x + 1$ and $dx = du/2$. Many people included steps with incorrect notation such as $\int_{0.5}^{2} u^{1/2}\, du/2$ (it should be $\int_2^5$). In this case, I gave credit if it led to the right answer, but be careful with this, since the work matters too.

2c) $9\pi/4$ using the area of part of a circle of radius 3.

2d) $-|8-x|$ using FTC#2, with a minus sign from reversing the $\int_x^2$. It is a bad idea to use FTC # 1 (and the abc thm, etc), though that might work out with a lot of care.

2e) $\tan^{-1}|_0^1 = \pi/4$. This uses FTC #1, of course, but the common fails were of memory, that $\frac{d}{dx}\tan^{-1} = \frac{1}{x^2+1}$ and $\tan(\pi/4) = 1$, etc.

3a) The final answer is $s(t) = 256 - 16t^2$, for approx 1 point, but as explained in class and on the exam, you need to derive this for full credit (not just memorization of a physics formula). Start with $v(t) = \int -32\, dt = -32t + C$, etc.

3b) 4 seconds.

4) 51 / 4 = 12.75. There are several styles, but my preference is to first list the $x_k^* = x_k = 3/2, 2, 5/2$ and 3. This seems easy, but if necessary, you can use formulas like $x_k = a + k\Delta x$ instead, to get this list. Then $f(x_1) = (3/2)^2 + 1 = 9/4 + 1$, etc. Adding these 4, then multiplying by $\Delta x = 1/2$, we get a Riemann sum of $51/4$. Of course, decimals are also OK if you prefer.

It is OK to use Σ notation and the summation formulas instead. But in my opinion, that style is overkill for a simple problem with $n = 4$. Generally instructions to "estimate" something require a simplified numerical answer such as $51/4$, so you should not leave Σ in your final answer. For this class, a partially-simplified answer such as $102/8$ is usually OK.

5) $31\pi/5$.

6) TFTFT

7) See the text or the lectures. As usual, the grades depended a lot on the explanations. Some people gave a calculation with no words attached, and others explained each step well. In general, you should state the theorem (and any assumptions) before proving it.

Bonus) $2^\pi = e^{\pi \ln 2}$. For full credit, say more. For example, 'In effect, this means $\ln(2^\pi) = \pi \ln 2$ since e^x is the inverse function of $\ln(x)$. And $\ln(x)$ has been defined as $\int_1^x \frac{dt}{t}$.'