MAC 2312 Noon, Jan 30, 2018
Exam I Key Prof. S. Hudson

Short answer: 7pts each. If any of these do not exist, explain.

1) $\sum_{k=1}^{30}\ k(k-2)$

2) $\int_0^{\pi/4} \sec^2 x\ dx$

3) $\int_0^{2\pi} |\sin(x)|\ dx$

4) $\int_0^3 x\sqrt{1+x}\ dx$

5) $\int_{-1}^1 \frac{dx}{1+x^2}$

6) $\int_{-1}^1 \sqrt{1-x^2}\ dx$

7) $\frac{d}{dx}\int_7^x \ln(t)\ dt$

8) [10pts] A particle moves with velocity $v(t) = e^{-t}$ for $0 \le t \le 4$. Find the average velocity of the particle during this period.

9) [10pts] Compute $\sum_{k=1}^n f(x_k^*)\Delta x_k$ using the Left Endpoint Rule, given that
$f(x) = x+2$, $a = 0$, $b = 5$, $n = 3$.
$\Delta x_1 = 1$, $\Delta x_2 = 2$ and Δx_3 is not given.
You should determine Δx_3 and the x_k^*, etc, if needed.

10) [8 pts] A car stopped at a toll booth leaves at time $t = 0$ with a constant acceleration of $a = 8$ ft/sec^2. At the time it leaves the booth it is 350 feet behind a truck traveling with a constant velocity of 5 feet per second. How long will it take for the car to catch the truck ? Simplify if possible. If you use the quadratic formula, one of these may help: $55^2 = 3025$, $65^2 = 4225$, $75^2 = 5625$.

11) [15pts] Answer True or False to each part. You do not have to explain.

The average value of the numbers $\{1, 2, 3 \ldots 1000\}$ is 500.
The average value of $f(x) = 10x - 7$ on $[1, 7]$ is equal to $f(4)$.
For any partition P of the interval $[1, 2]$ into n subintervals, $\sum_{k=1}^n \Delta_k = 1$.
The displacement of a particle in rectilinear motion is always nonnegative.
$\int_0^{\pi/2} \sin^{30} x\ \cos x\ dx = \int_0^1 u^{30}\ du$.

12) [8 pts] Choose ONE proof. Use sentences and formulas with complete justifications. The grading will be based on the clarity of your logic and explanations, as much as on any calculations involved. If you use the back of the page, leave a note here.

A. State and prove the FTC #2 (about $\frac{d}{dx}\int^x$).

B. State and prove the formula for $S = \sum_{k=0}^n ar^k$.

Bonus: (5 points, maybe hard): Calvin and Hobbes each compute a Riemann sum for $\int_0^\pi \sin(x)\ dx$, both using $n = 4$. Calvin gets 1.0 and Hobbes gets 5.0. Can both of these be correct ? Explain.

Remarks and Answers: The average among the top 24 was about 65, which is fairly normal. The two highest scores were 94 and 90. The grades were similar on all the problems, but slightly lower on problem 9 (50%). The unofficial scale for this exam is:

A's 73 to 100
B's 63 to 72
C's 53 to 62
D's 43 to 52

1) Use the distributive property, split it, and get $\frac{30\cdot 31\cdot 61}{6} - 30\cdot 31$.

Your grade for problem 1 is in the left margin. You may also see a note nearby, such as "$\sum = 14$", which you can ignore; it is just to help me add the grades.

2) $\tan(x)|_0^{\pi/4} = 1$.

3) Use the abc method to get 4. It is also OK (and equivalent) to sketch a graph of the function and add the areas of the two bumps, 2+2=4. The graph is entirely above the x-axis, since $|\sin(x)| \geq 0$. So, as a 'sanity check', the answer must be positive. For this reason, I gave less partial credit than usual for 'crazy' answers like -4.

4) With $u = 1 + x$, get $\int_1^4 (u-1)u^{1/2}\ du = \int_1^4 u^{3/2} - u^{1/2}\ du = \frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}|_1^4 = \frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3} = \frac{62}{5} - \frac{14}{3}$.

The substitution $u = 1 + x$ is maybe not 100% obvious, but it is often a good idea to let u be some formula in a square root or in a denominator - or in any composite function you see. Integration often requires some trial and error, and practice helps.

It is wrong (slightly wrong?) to start with $\int_0^3 (u-1)u^{1/2}\ du$. Change the limits of integration as soon as you switch from x to u. A few people started this way and made adjustments later. That should lead the right answer, but it is not a good practice.

5) $\tan^{-1} x|_{-1}^{1} = \pi/2$. It might help to notice that $\tan x$ and its inverse function are both odd. By definition, $\tan^{-1}(-1) = -\pi/4$, not $3\pi/4$.

6) $\pi/2$ from half of $\pi r^2|_{r=1}$. It is not a quarter circle (check the domain).

7) $\ln(x)$ by FTC 2.

8) By the definition of average value of $v(t)$, $\frac{\int_0^4 e^{-t}\ dt}{4} = \frac{1-e^{-4}}{4}$. Note that $\int e^{-t} = -e^{-t}$.

The Calculus I approach, $\frac{\Delta s(t)}{\Delta t}$, should lead to the same answer, but requires a few more steps. Some people used $\frac{\Delta v(t)}{\Delta t} = \frac{e^{-4}-1}{4}$ by mistake (this would be average acceleration).

Some people wrote $\frac{e^{-4}-1}{4}$ showing no work and did not get much credit. It is just possible that they did everything right in their head, but missed a minus sign. For good partial credit, show your work !

9) $2 \cdot 1 + 3 \cdot 2 + 5 \cdot 2 = 18$ (but showing work). A common mistake was to start with $\Delta x = (b-1)/n = 5/3$. That formula is used only with *regular* partitions. Here, you are told that $\Delta_2 \neq \Delta_1$, so P is not regular. The calculations will be pretty primitive. In fact, a simple picture of 3 rectangles is a pretty good guide to solving this problem. The areas are 2, 6 and 10.

If you are asked for an exact answer for area, you will probably be allowed to use a regular partition, and you should, and then you can use $\Delta x = (b-1)/n$ etc.

10) 10 sec. Compute the two position functions and set them equal; $4t^2 = 350 + 5t$. Solve for t, probably using the quadratic formula, $t = -b \pm \sqrt{b^2 - 4ac}/2a$. If you forget this formula you can get the same result by completing the square, but it's better to learn the formula.

11) FTTFT

12) See the text. Most people chose 12A and wrote something similar to the proof in the lecture. Common mistakes:

a) The problem asks you to *state the theorem*. Normally this should be done at the start and should include any hypotheses like "If f is continuous on $[a, b]$" (etc). But you can not assume f has an antiderivative F. It is not part of FTC.2. This would make the proof easy, but it is hard to justify without using FTC.2 itself, which is circular.

b) Too many notation problems, such as over-using $\Rightarrow$. This normally means "implies" and is stuck between two sentences. So, $3 + 4 = 7$ makes sense, but $3 + 4 \Rightarrow 7$ does not. Also, in the definition-of-derivative step, some people used $f(x + h)$ in the numerator, but in this context you need $\int_a^{x+h} f(t)\, dt$ instead. Third, the abbreviation $\int^x$ is pretty bad. I didn't take off points, since I used that in the problem (but only to jog your memory about FTC.2).

c) Some answers included various bits of the proof but not in the correct order, or with major gaps in between. Probably this resulted from memorizing the proof imperfectly. Before the exam, work mainly on understanding the proofs, keeping any memorization to a minimum.

Bonus) No. Calvin's answer is a possible R.Sum, but Hobbes's answer is not. The largest possible R.Sum is π since every $f(x_k^*) \leq \sin(\pi/2) = 1$ and $b - a = \pi$. For full credit, include the comment about π, but maybe not $f(x_k^*)$.

A few people wrote that two different answers are not possible. This is correct for computing an integral such as $\int_0^\pi \sin(x)\, dx$, but it is not true for R.Sums, which are only approximations, with some flexibility in their calculations.