Exam 1 +Key | Jan 30, 2020
MAC 2312 | S Hudson

1) [5pts] State the main trig identity used in evaluating $\int \cos^2 x \, dx$.

2) [5pts] Let $y = \int_7^x \sqrt{1+t^2} \, dt$. Find $\frac{dy}{dx}$.

3) [7pts] Sketch a graph of $f(x) = 2 - |x|$ for $-1 \le x \le 1$. Use known area formulas to evaluate $\int_{-1}^{1} 2 - |x| \, dx$.

4) [8pts] Evaluate $\int \sec 2x \tan 2x \, dx$ using a u-substitution.

5) [10pts] Evaluate $\int_{-1}^{0} \sqrt{y+1} \, dy$ using a u-substitution. For full credit, use the Ch. 5.6 style; convert it to a definite integral (ending with "du") with new limits of integration, etc.

6) [10pts] Sketch the two curves $y = x^2$ and $y = 8x$ together in the same plane. Find where they intersect. Find the area of the region that they enclose.

7) [10pts] Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{9 - x^2}$ and $y = 0$ about the x-axis. Use a definite integral as in Ch. 6.1 (not just a formula from geometry).

8) [10pts] Choose ONE of these application problems, circle it, and set it up as an integral. You do not have to evaluate the integral.

a) Sketch the region bounded by $y = x$ $y = -x/2$ and $x = 2$. Use the Shell method to find the volume of the solid obtained by revolving this region about the y-axis (just write out the integral).

b) Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$. Just write out the integral.

9) [10pts] Calculate the exact area under the graph of $f(x) = x^2 + 3$ over the interval $[0, 2]$ using Riemann sums as follows. Let R_n be the Riemann sum obtained using n equal subintervals of [0,2] and the right-endpoint rule. Find a formula for R_n and then take a limit as $n \to \infty$ to compute the area. Option: you can replace $f(x)$ above with the slightly simpler function $f(x) = x + 1$ for partial credit, but follow the same rules.

10) [15pts] Answer each with True or False. You do not have to explain.

If S is a Riemann sum for $\int_0^3 x^2 \, dx$ then $0 \le S \le 100$.

If S is a Riemann sum for $\int_0^3 x^{-1} \, dx$ then $0 \le S \le 100$ (Notice that x^{-1} may not be integrable on [0,3]).

If P is a regular partition of $[1, 3]$ with $n = 20$ then $x_2 = 1.2$.

The average value of $f(x) = 1 + \sqrt{1 + \sqrt{x}}$ on $[2, 4]$ is at least 2.

In Ch.6.3 the arc length formula is $L = \int_a^b (1 + f'(x)^2) \, dx$.

11) [10pts] Choose ONE proof. If possible, use sentences or formulas with complete justifications. Recall that the grading will be based on the clarity of your logic and explanations, as much as on any calculations involved.

A. State and prove the FTC Part 1 (about $\frac{d}{dx}\int f$). For a little extra credit include the steps that use the M.V.Thm.

B. State and prove the Mean Value Theorem for integrals (Thm 3, page 330). It is OK to quote prior theorems about averages and intermediates values.

C. State and prove the formula for $\sum_{k=1}^{n} k$, sometimes called the Little Gauss summation formula.

Bonus [5pts]: Compute this sum and simplify it completely to an irreducible fraction.

$$\sum_{k=0}^{29} \left(\frac{10 + 4k/3}{200 + 27k} - \frac{10 + 4(k+1)/3}{200 + 27(k+1)} \right)$$

Remarks and Answers: The average among the top 30 students was approx 63 / 100, with high scores of 92 and 84, which is a normal result. The best scores were on problems 2 and 6 (75%) and especially on 5 (90%). The worst scores were on problems 1 (45%) and 9 (30%). I expected higher scores on 1, since it is precalculus and I have emphasized its rising importance a few times already. Here is an unofficial advisory scale for Exam I:

A's - 74 to 100
B's - 63 to 73
C's - 52 to 62
D's - 41 to 51

This scale is more accurate than the one on the syllabus. Now the answers:

1) $\cos^2 x = \frac{1+\cos(2x)}{2}$. If you have not memorized this pre-calculus identity yet, do that soon. It is one of the most useful ones later in MAC 2312 and 2313. You can check it by setting $x = 0$ etc, which may help you remember it.

2) $\sqrt{1+x^2}$ by the FTC, Part One. Just change the t to x, though some of the homework examples are a bit harder.

3) The graph looks like a house, which you can split into two trapezoids (or other shapes). The total area, and the final answer, is $3/2 + 3/2 = 3$.

Without the instructions to use area, you could skip the picture and use $\int_{-1}^{0} 2 + x\, dx + \int_{0}^{+1} 2 - x\, dx = 3/2 + 3/2 = 3$. This is based on $2 - |x| = 2 \pm x$ (and a little more thought). Splitting an integral like this is a good way to remove absolute values. But for this example, area is simpler.

4) Using $u = 2x$, get $\frac{1}{2}\sec(2x) + C$. While you are learning, u-substitutions may involve some trial and error, but since $2x$ appears inside another function, inside parentheses, try

$u = 2x$, which quickly works. Several people tried $u = \tan 2x$, but the du calculation doesn't work well; the derivative of tan is $\sec^2$, which does not quite match the sec in the original integral. Learn to use trial and error, and expect to get better with practice.

5) 2/3. Set $u = y + 1$, and get $\int_0^1 u^{1/2}\ du = \frac{2}{3}u^{3/2}|_0^1 = \frac{2}{3}$. Several people got to $\frac{2}{3}u^{3/2}|_0^1$ but then switched back to $(y+1)^{3/2}$ notation, and got the wrong answer.

Switch back to y notation with indefinite integrals (in Ch 5.5) because you don't want u in your final answer. Don't switch back with definite integrals, like this Ch 5.6 example. If you insist on switching back in this question, use $\frac{2}{3}(y+1)^{3/2}|_{-1}^{0} = \frac{2}{3}$, but that is an unusual and inefficient style.

6) The line is above the parabola, over the interval $0 \le x \le 8$. Area $= \int_0^8 8x - x^2\ dx = \cdots = 256/3$. I was mainly looking for the correct final answer, 256/3. The intersection and the sketch help you get there (for example, to avoid common mistakes like $\int_0^8 x^2 - 8x\ dx$).

7) 36π. The graph, highly recommended, is a semicircle and the solid is a sphere of radius 3. Use the disk method, $V = \pi \int_{-3}^{3} \sqrt{9-x^2}^2\ dx = \cdots = 36\pi$. You can check this using $V = \frac{4\pi r^3}{3}|_{r=3} = 36\pi$ (the geometry formula derived using this same MAC 2312 method). We did all this in class.

8a) $V = \int_0^2 2\pi x(3x/2)\ dx = \int_0^2 2\pi 3x^2/2\ dx$.

8b) $L = \int_0^4 \sqrt{1 + (3x^{1/2}/2)^2}\ dx = \int_0^4 \sqrt{1 + 9x/4}\ dx$. I feel both answers should be simplified slightly as I have done here, but I did not deduct points this time.

9) $R_n = \sum_{k=1}^{n}((2k/n)^2+3)(2/n) = 8/n^3 \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 6/n = 8(n+1)(2n+1)/6n^2 + 6$, with some steps omitted. The final answer is the limit, 26/3. No credit for using an integral to get this number. A little partial credit for a good start, with $\Delta x = 2/n$ and $x_k = 2k/n$, etc. See exercises 5.3.43, etc.

10) TFTTF. See me to discuss these.

11) Parts A and B are in the book and the lectures and my Exam I proof list. Part C is a little easier and was given in a lecture. Common mistakes were to give only the statement or only the proof, or to offer an example instead of a proof. I assume this was from insufficient preparation.

Bonus) In honor of the first month of the new decade, the sum is 1/2020. The problem telescopes to $\frac{10+4k/3}{200+27k}|_{30}^{0}$, which eventually simplifies to 1/2020. A few people came very close and one person got this exactly right.

By the way, this was written on 02/02/2020, the first palindrome date (if using day/month/year) in 909 years.