Exam III — Nov 6, 2014
MAC 2312 — S Hudson

There were two versions of this exam, but they were very similar, differing only in some of the constants in problems 1c, 1d and 4b. The exam appearing below is Version A. See the answers for more about B.

1) [8 points each] Compute each integral.

1a) $\int \tan^{-1}(x)\ dx$

1b) $\int \sec^3 x\ \ \tan x\ dx$

1c) $\int \sin(2x)\cos(3x)\ dx$ Hint: if you have forgotten this type, playing around with the identity for $\sin(a+b)$ might help.

1d) $\int \frac{dx}{x^2+6x+10}$

1e) $\int \frac{x^3}{x^2-1}\ dx$

1f) Simplify this to a single fraction. For partial credit, at least show how to handle an improper integral properly.

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$$

2) [12pts] a) Estimate $\int_1^7 x^2\ dx$ using T_3 (the Trapezoid Rule with $n=3$) and simplify as far as possible. Be careful, since calculation errors may affect later parts of this problem.

b) Compute the maximum possible error using the usual bound for E_T. If you do not remember the formula, you can use this (incorrect) one for partial credit: $|E_T| \le \frac{(b-a)^2 K_2}{8n^3}$.

c) Compute E_T exactly and comment on whether it exceeds the bound you got in 2b.

3) [20 pts] Answer True or False:

If $\sum a_k$ converges then $\lim a_k = 0$.
$\int_1^\infty x^p\ dx$ converges if and only if $p < -1$.
$\int_0^1 x^p\ dx$ converges if and only if $p \ge 0$.
The sequence $5(-1)^k + k^2$ is eventually monotonic.
The sequence $3(-1)^k - k$ is eventually monotonic.
If a sequence is eventually decreasing, then it is bounded above.
If a sequence is bounded and eventually decreasing, then it converges.
If $\sum a_k$ converges, then its partial sums are bounded.

Simpson's Rule on $\int_0^5 x^4\ dx$ with $n = 10$ produces an exact answer (so $E_S = 0$).
The Midpoint Rule on $\int_0^5 x^4\ dx$ with $n = 10$ gives an overestimate, $M_{10} \geq \int_0^5 x^4\ dx$.

4) [5pts each] Decide whether each series converges. Justify each answer by showing any work needed (and probably mentioning a test, a theorem or some other reasoning). You do not have to compute the sums.

a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

b) $\sum_{k=1}^{\infty} \frac{(-2)^{k+1}}{6^{k-1}}$

c) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2}$

d) $\sum_{k=1}^{\infty} (k^3 - (k+1)^3)$

Bonus [about 5 points]: Compute $\sum_{k=1}^{\infty} \frac{2}{k^2+2k}$ and simplify.

Remarks + Scales: The average among the top 20 was 58 out of 100, with high scores of 72 and 71. This is a bit lower than Exams I and II and a bit lower than expected. The best scores were on 1b (90%) and the worst were on 1c) and 1f) (approx 40 to 45% each). The scale for Exam III is:

A's 70 to 100
B's 60 to 69
C's 50 to 59
D's 40 to 49

The average for the semester (based on the three exams so far, but not on HW or EC) is approx 59. I have written yours on the upper right corner of your Exam III, and you should check that it matches your records. You can use the scale above for that, but adjust it approx +1 point (59-58=1), so that A's start at 71, etc.

Answers:

1a) $x \tan^{-1}(x) - \frac{\ln(x^2+1)}{2} + C$, from IBP.

1b) $\frac{1}{3}\sec^3(x) + C$, from setting $u = \sec(x)$.

1c) There were two versions. In both, use $\sin(a)\cos(b) = (\sin(a+b) + \sin(a-b))/2$, which you should be able to get from the famous $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ identity, if necessary.

Version A, with $\int \sin(2x)\cos(3x)\ dx$. Answer $= \frac{1}{2}[\frac{-\cos(5x)}{5} + \cos(x)] + C$

Version B, with $\int \sin(2x)\cos(4x)\ dx$. Answer $= \frac{1}{2}[\frac{-\cos(6x)}{6} + \frac{\cos(2x)}{2}] + C$

In problems like this one, there are sometimes alternative approaches, such as $\cos(4x) = \cos^2(2x) - \sin^2(2x)$, which might lead to an equivalent answer (though appearing to be different). But I only noticed this working out well once.

1d) In both versions, notice the denominator does not factor (so forget about partial fractions). Complete the square, followed by a substitution (such as $u = x + 3$ or $\tan\theta = x + 3$ in Version A).

Version A, with $\int \frac{dx}{x^2+6x+10}$. Answer $= \tan^{-1}(x+3) + C$,

Version B, with $\int \frac{dx}{x^2+4x+5}$. Answer $= \tan^{-1}(x+2) + C$,

1e) Division and then $u = x^2 - 1$ (or partial fractions) leads to $\int x + \frac{x}{x^2-1}\ dx =$
$\frac{1}{2}[x^2 + \ln|x^2 - 1|] + C$.

Two unusual ideas are also possible: 1) Set $u = x^2 - 1$ from the start. A clever shortcut, but not easy to find, and nobody carried this out correctly. Or, 2) set $x = \sec\theta$. This leads to some messy trig, which nobody got through correctly, so this is probably not best.

1f) $\lim_{M\to\infty} \sec^{-1}(M) - \sec^{-1}(2) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$. Note that $\sec^{-1}$ is similar to $\tan^{-1}$ at infinity. Also recall $\cos(\pi/3) = 1/2$, $\sec(\pi/3) = 2$, etc.

Each part of Problem 2 was worth 4 points.

2a) $\frac{6}{2\cdot 3}(1 + 2\cdot 9 + 2\cdot 25 + 49) = 118$. If you can't remember the standard formula, it might be safer to simply add the areas of three trapezoids. Nobody did this, and it doesn't quite fit the instructions, but I probably would've given full credit.

2b) $K_2 = 2$ and $\frac{(b-a)^3 K_2}{12n^2} = 4$. You got approx 2 points out of 4 for using the incorrect formula correctly.

2c) $\int_1^7 x^2\ dx = 342/3 = 114$ so $|E_T| = 118 - 114 = 4$. It matches the upper bound *exactly.* This is pretty unusual, and not to be expected. I think it happens only when $f(x)$ is a parabola, but further discussion of that is beyond the level of our course.

3) TTFTF TTTFF

4a) Converges (C). This is the Alt.Harm.Series (which is well-known). It is also OK to say "by the A.S.Test" but then you should at least mention that $1/k$ decreases to zero. It is NOT OK to quote the Root Test (etc) which requires $a_k \geq 0$. It is NOT OK to use the RTAC, since $\rho = 1$ is inconclusive.

Note on grading. If you did not write Converges or "C", you got zero credit, regardless of your reasoning. This part of the problem is like True-False. If you wrote Converges, but gave no reason, or a completely incorrect reason, you got 2 out of 5. I often gave partial credit (3-4 points) for *slightly* flawed or incomplete reasoning.

4b) C. This is geometric with $|r| = |-2/6| < 1$. Or, in Version B of the exam, $|r| = |-2/3| < 1$, which is very similar. Again, you cannot use the Root Test etc, unless you also use absolute convergence, but that "uses an elephant gun on a mosquito".

4c) C. This is a p-series with $p = 1.5 > 1$.

4d) D. Since this is telescoping, we get $s_n = 1 - (n+1)^3 \to -\infty$. By definition of convergence, this series does not.

It is also OK to apply the Div Test, if you worked out that $\lim a_k = -\infty$. The Integral Test is possible, but it's non-obvious and harder to explain. If you 'had the right idea', but did not mention the definition or a specific Test, your answer was probably not good enough for full credit. Try to be as clear and specific as possible.

Bonus) Use partial fractions to change this to a telescoping series, in which two early terms don't cancel, $\sum(\frac{1}{k} - \frac{1}{k+2}) = 1 + 1/2 = 1.5$