Exam III and Key | Mar 31, 2016
MAC 2312 | S Hudson

1) [8 pts each] Solve each, or show it diverges. Use a limit with any improper integrals. Include C's, as needed.

$\int \ln(2x)\ dx$

$\int_0^{\pi/6} \sin(2x)\cos(4x)\ dx$.
If you are stuck, one of these may help: $\sin^2(x) = \frac{1-\cos(2x)}{2}$ and $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$.

$\int \frac{dx}{\sqrt{x^2-1}}$

$\int \frac{x^2\ dx}{(x+2)^3}$

$\int_0^9 \frac{dx}{\sqrt{9-x}}$

2) [12 pts] 2a) Use the Trapezoid Rule with $n = 4$ to approximate $\int_{-1}^{1} \frac{dx}{1+x^2}$. Simplify your answer. Notes for 2a and 2b: $f(-1) = 0.5$, $f(0.5) = 0.8$ and $f(2) = 0.2$. Also, $f''(x)$ increases from -2 to -1 on the interval $[0, 1]$. You may not need all this info, and you may need to compute a few other numbers yourself.

2b) Recall that $|E_T| \leq \frac{(b-a)^3 K_2}{12n^2}$. How large must n be so that the error when using T_n is less than 3/400 ?

3) [10 pts] In 3a and 3b (two separate problems), $\sum_{k=1}^{\infty} a_k$ has partial sums $\{s_n\}$:

3a) Given that $a_k = (1/3)^k$ compute s_2. Does $\{s_n\}$ converge ? Explain.

3b) Given that $s_n = \frac{2n}{3n+1}$ compute a_3. Does $\sum_{k=1}^{\infty} a_k$ converge ? Explain. Does $\{a_k\}$ converge ? Explain.

4) [8 pts] Prove that $a_n = n^2 - 6n - 7$ is eventually strictly monotone.

5) (10 pts) Compute each sum.

$\sum_{k=1}^{\infty} (\frac{1}{k+2} - \frac{1}{k+4}) =$

$\sum_{k=3}^{\infty} \frac{6}{4^k} =$ Notice that this starts at $k = 3$.

6) (10pts) Answer True or False:

The series $2 - 1 - 1 + 2 - 1 - 1 + 2 - 1 - 1 \ldots$ converges to 0.

$\lim_{n\to\infty} \cos(n\pi) = 1$.

Every increasing sequence that is bounded above converges.

If $a_0 = 2$ and $a_{n+1} = 5 - a_n, \forall n \geq 1$, then $a_{2016} = a_{1944}$.

The harmonic series converges to 0.

7) (10 pts) Choose ONE proof, explain thoroughly:

a) State and prove the Comparison Test. You can omit part b, the contrapositive part.

b) State and prove the Divergence Test. You can omit the inconclusive case.

BONUS: (5 pts) Give an example of a sequence of positive numbers $\{a_k\}$ with $\lim a_k = 0$ that is not eventually monotone. For maximum credit, give a specific formula.

Remarks and Answers: The average among the top 20 scores was 46%, which is unusually low, perhaps reflecting a hard exam. The highest scores were 90 and 89. There was only one score between 48 and 74. The results were OK on problems 1a and 6, but pretty low on 1e and 4, about 27% on each. Here is the unofficial scale for the exam:

A's 60 to 100
B's 50 to 59
C's 40 to 49
D's 30 to 39

To estimate your semester grade, you can average your letter grades for the 3 exams so far, but the scale below is probably more accurate, based on a numerical average of your 3 grades. The average average is about 61. I do not have enough info yet to include HW, EC, etc into this.

A's 69 to 100
B's 59 to 68
C's 49 to 58
D's 39 to 48

Remarks on 1): An answer like 'Diverges' only makes sense for improper integrals. Only part 1e is improper, and it does not diverge. People also wrote 'Converges' for some answers, but again that word only really makes sense for improper integrals. And, if it does converge, you should give a specific numerical answer instead. On the *next* test, I may ask you to classify a series, and then an answer like 'Diverges' may be correct.

Each of the five parts is graded in black ink, with the total written in the margin, in red. If you want to check my addition for your total score on the exam, only count the numbers in the margins. They should already include any other numbers on your exam.

1a) $x\ln(2x) - x + C$ using IBP. It might be easiest to start with $\ln(2x) = \ln(2) + \ln(x)$, or with $u = 2x$, but these steps are not required.

1b) 1/24, from integrating $\sin(6x)$ and $\sin(2x)$.

1c) $\ln|x + \sqrt{x^2 - 1}| + C$. Start with $x = \sec\theta$ and $dx = \sec\theta\tan\theta\ d\theta$.

1d) $\ln|x+2| + 4(x+2)^{-1} - 2(x+2)^{-2} + C$. Partial fractions is the expected method, but 1 or 2 people used $u = x+2$ and $x^2 = u^2 - 4u + 4$ successfully, and that is actually a bit easier.

1e) 6. But this is improper, so for full credit, you had to start with something like $\lim_{M\to 9^-}\int_0^M \frac{dx}{\sqrt{9-x}}$ (see the instructions for Problem 1, but this is the standard method).

I don't recall discussing u-subs for improper integrals. It is OK to start with $u = 9-x$, but this leads to another improper integral of about the same difficulty. People tried various trig subs, but failed. I don't recall anyone trying $x = 9\sin^2(\theta)$, which is not a standard idea, but it should work.

2a) Parts 2a and 2b were 5 points each. $(\frac{2}{8})6.2 = 1.55$.

2b) $n = 14$, from $K_2 = 2$ and $\frac{16}{12n^2} \leq \frac{3}{400}$ (so $n^2 \geq 1600/9$ and $n \geq 40/3$). Many people wrote $\geq$ instead of $\leq$, did not get K_2, or got lost in the arithmetic. I probably meant to use 4/300, which would have worked out just a little nicer.

3a) [4 pts] $s_2 = 4/9$. Since the series is geometric with $r = 1/3 < 1$, the series converges, so $\{s_n\}$ converges too.

Try to be precise and complete. Avoid ambiguous phrases, such as 'it' converges. You needed to at least mention $\{s_n\}$ in your explanation for full credit.

3b) [6 pts] $a_3 = 1/35$. Many people confused a_3 with s_3, and this confusion often carried over into the next two parts of 3b. Since $\lim s_n = 2/3$ converges, $\sum_{k=1}^{\infty} a_k$ also converges, by definition. Since $\sum_{k=1}^{\infty} a_k$ converges, $\lim a_k = 0$ (so $\{a_k\}$ converges) by the Divergence Test.

4) Many people decided correctly that the sequence is eventually increasing, but few offered any real proofs, with the specific inequalities needed. There are several fairly simple methods, such as one of these:

a) Let $f(x) = x^2 - 6x - 7$ so that $f'(x) = 2x - 6 > 0$ for $x \geq 3$. Since $f(x)$ strictly increases on $[3, \infty)$, a_n strictly increases for $n > 3$.

b) $a_{n+1} - a_n = [(n+1)^2 - 6(n+1) - 7] - [n^2 - 6n - 7] = 2n - 5 > 0$ for $n \geq 3$. This shows that eventually $a_{n+1} > a_n$, thus strictly increasing.

c) In principle, you could show that $a_{n+1}/a_n > 1$ for $n \geq 3$. But that algebra is a bit harder than option b.

5) 7/12 and 1/8

6) FFTTF

7) See the text or lecture notes. Most people included enough words, probably better than usual. But in a proof, it is not enough to vaguely state the main idea. The reasoning should be airtight, usually including a few calculations, usually requiring some care with notation. For (a), some people wrote 'If the bigger sequence converges' (etc). Complaints: 1) the bigger sequence deserves a name, such as $\{b_k\}$; 2) it is the *series*, $\sum b_k$, that converges. Try not to confuse functions, sequences, series, sets, etc. Also, very few people stated clearly that the partial sums of $\sum a_k$ are monotonic and bounded, which are two key ideas of the proof.

B) There are many examples, but they are not easy to find. I saw only one correct answer, $(1 + \frac{(-1)^k}{2})/k$. It is fairly obvious that this example is positive and approaches zero. If you write out some terms, you will see that it is not eventually monotone. A proof of that was not required.