Exam III — June 17, 2016
MAC 2312 — S Hudson

You can use the Ch.7.6 error bounds $|E_M| \le \frac{(b-a)^3 K_2}{24n^2}$ and $|E_T| \le \frac{(b-a)^3 K_2}{12n^2}$ for any problems on this exam.

1) [10 points total] Note the formulas above.

1a) Find n, as small as you can, so that the midpoint approximation M_n for $\int_0^2 5x^3\,dx$ is accurate to 3 decimal places. You should not need a calculator if you know that $(300)^2 = 90,000$, etc. Justify briefly (unless your work is already very clear).

1b) For the same problem as 1a), will M_n be greater than, less than, or equal to $\int_0^2 x^3\,dx$? Explain briefly.

2) [10 points] Determine whether this improper integral converges or not. If it converges, compute it using the methods of Ch.7.7 (without shortcuts), $\int_{-1}^{1} x^{-2/3}\,dx$.

3) [10 points] Compute $\int \frac{2x^2-9x-9}{x^3-9x}\,dx$.

4) [10 points] Show that $a_n = \frac{n}{e^n}$ is eventually strictly monotone.

5) [10 points total] Compute each sum, or explain why the series diverges:

a) $\sum_{k=0}^{\infty}(\frac{1}{k+2} - \frac{1}{k+1})$

b) $\sum_{k=0}^{\infty}(\sqrt{k+1} - \sqrt{k})$

6) [10 points] Find the general term of the sequence $\frac{1}{3}, \frac{-1}{9}, \frac{1}{27}, \frac{-1}{81} \ldots$. Determine whether the sequence converges, and if so, find its limit.

7) [10 points total] Determine whether the series converges, with brief justification.

a) $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$

b) $\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$

8) [20 pts] Answer True or False: you do not have to explain.

The sequence $a_n = \frac{n^2+8n+1}{n^3+1}$ is eventually monotone.

The alternating harmonic series can be rearranged to converge to 7.

The alternating harmonic series converges absolutely.

Every monotone sequence that is bounded above converges.

If $a_0 = 1$ and $a_{n+1} = 5 - a_n, \forall n \ge 1$, then $a_{2016} = a_{2012}$.

T_{100} always gives a smaller error than M_4.

The improper integral $\int_0^\infty x^p\ dx$ diverges for all $p > 0$.

The series $\sum_{k=1}^\infty k^{-p}$ diverges if $0.5 < p < 1.5$.

The Fibonacci sequence $\{f_n\}$ diverges but the sequence $\{\frac{f_{n+1}}{f_n}\}$ converges.

The partial sums of $\sum_{k=1}^\infty \frac{3}{k^2+2}$ are bounded.

9) [10 pts] Compute $\int \frac{x^2-8}{x+3}\ dx$

Bonus) The Gamma function is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\ dt$ for $x > 0$. Compute $\Gamma(10)$. You can answer in terms of a more basic function rather than an explcit number (for example, an answer such as 5^{10} would be OK).

Remarks and Answers: The average among the top 22 was 65% with two high scores of 82 each. The averages were under 50% on problems 1 and 4, but approx 90% on 9. Here is a scale for the exam, the same scale as for Exam II:

A's 72 to 100
B's 62 to 71
C's 52 to 61
D's 42 to 51

Current scale for the semester, 3 exams only:

A's 73 to 100
B's 63 to 72
C's 53 to 62
D's 43 to 52

1a) $n = 201$. Use $\frac{2^3 \cdot 60}{24 \cdot n^2} < 0.0005$ so that $n^2 > 40,000$. There were many minor mistakes on this one, such as $K_2 = 30x = 30$, but you should set $x = 2$, to maximize $30x$, and get $K_2 = 60$.

1b) M_n is smaller than the integral because x^3 is concave up (see lecture notes).

2) It converges to 6. Since there is a VA at $x = 0$, use 'abc' and $\lim_{m\to 0^-} \int_{-1}^{m} x^{-2/3}\ dx = \cdots = 3$, etc.

3) $\ln|x| + 2\ln|x+3| - \ln|x-3| + C$ from partial fractions.

4) It is eventually *decreasing*. The two most common explanations were these:

a) $\frac{a_{n+1}}{a_n} = \frac{n+1}{n\ e} < 1$ if $n \geq 1$, because $\frac{n+1}{n} = 1 + \frac{1}{n} \leq 2 < e$. You should include all of this explanation (or something equivalent) and of course it is OK to say more.

b) Let $f(x) = xe^{-x}$ and get $f'(x) = [1-x]e^{-x} < 0$ for $x > 1$. This implies $a_n < a_{n+1}$

for $n > 1$. In my opinion, the step $f(x) = xe^{-x}$ is needed, because mixing derivatives with the letter 'n' is bad notation.

5a) -1. It is telescoping and $s_n = -1 + \frac{1}{n+2} \to -1$.

5b) D, similar to 5a, but $s_n = \sqrt{n+1} \to \infty$.

6) $a_k = (-1)^{k+1}3^k$ for $k = 1, 2, \ldots$, it converges to 0. Many people thought this was a problem about series, and made it harder than it was, though the word *sequence* is used twice in the problem. You do not need a convergence test for this.

7a) C, by the A.S.T. (and show a little work).

7b) D, by the Int.T (and show a little work). These two explanations are the only correct answers I recall. You cannot compare 7b to the H.S., for example, because the HS is bigger. You cannot use the A.C.T. in 7a (it does not CA).

8) TTFFT FTFTT

9) $x^2/3 - 3x + \ln|x+3| + C$. The rational function is improper, so divide. A few people used $x^2 - 8 = (x+3)(x-3) + 1$ instead, which is OK and leads to equivalent calculations. But division is more reliable.

B) 9! from IBP many times (or seeing the pattern earlier than that).