Exam III + Key | Noon, Apr 3, 2018
MAC 2312 | S Hudson

1) [7 points each = 49 total] Compute and simplify (within reason) the ones that converge. If any diverge, explain briefly.

a) $\int_e^{+\infty} \frac{dx}{x(\ln(x))^3} =$

b) $\int_{-\infty}^{+\infty} x\ dx =$

c) $\int_2^5 \frac{dx}{x(x^2-1)}$

d) $\int_{-3}^{-1} \frac{dx}{x^2+4x+5}$

e) $\sum_{k=1}^{\infty} 5(-2/3)^k$

f) $\sum_{k=1}^{\infty}(k+1)^3 - k^3$

g) For the previous example, compute the third partial sum, s_3.

2) [10 pts] A sequence is defined recursively by $x_1 = 2$ and $x_{n+1} = x_n/2 + 2$ for $n \geq 1$. You are given that every $x_n \leq 4$ (this is true and you do not have to check it).

2a) Compute x_3.

2b) Show that the sequence is monotone. There are several ways to do this; you might study $x_{n+1} - x_n$ or x_{n+1}/x_n, for example.

2c) Show that the sequence converges and find the limit, L.

3) [8 pts] A wall sitting on the x-axis goes from $x = 1$ to $x = 4$. We do not have an exact formula for its height $f(x)$, but know that $f(1) \approx 1.4$, $f(2) \approx 1.7$, $f(3) \approx 2$ and $f(4) \approx 2.2$.

3a) Use the Trapezoid Rule with $n = 3$ to approximate the area of the wall. Simplify your answer (to some number, like 3.8).

3b) Is your approximation in 3a) larger or smaller than the exact area ? Or, is this impossible to answer ? Explain.

4) [8 pts] Show that the series below converges using convergence test(s) from Ch 9.3-9.5. State clearly which test(s) you use, and show the required calculations.

$\sum_{k=1}^{\infty} \frac{2k}{k^3+1}$

5) (15pts) Answer True or False:

Every monotone sequence that is bounded above converges.

Eventually the partial sums s_n of the harmonic series exceed 1000.

Eventually the partial sums s_n of the p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ exceed 1000.

Eventually the partial sums s_n of $\sum_{k=1}^{\infty} (\frac{-1}{3})^k$ are monotonic.

If $a_{n+1} - a_n \geq 0$ for all n, then $\{a_n\}$ is an increasing sequence.

6) (10 pts) Choose ONE proof, explain thoroughly:

a) State and prove the Comparison Test (you can omit part b, the contrapositive part).

b) State and prove the Divergence Test.

BONUS: (5 pts) Give an example of a series $\sum_{k=1}^{\infty} a_k$ that can be classified as C or D with the Comparison Test but not with the Integral Test or the Ratio Test.

Remarks, Scales and Answers: The average among the top 24 scores was 52% with top scores of 83 and 65. This is rather low. The average scores were similar on most problems except on 1d, 1e and 2, which were below 35%. Here is an advisory scale for Exam III:

A's 64 to 100
B's 54 to 63
C's 44 to 53
D's 34 to 43

You can estimate your letter grade for the semester by averaging your three letter grades on the exams. Use the next scale:

A's 67 to 100
B's 57 to 66
C's 47 to 56
D's 37 to 46

1a) 1/2. The main issue here, and on 1b, was your *method* for handling an improper integral. For full credit, include a limit as $M \to \infty$ (and $u = \ln(x)$ and $\ln(e) = 1$).

1b) Diverges. For full credit, start with $\lim_{M\to+\infty} \int_0^M x\ dx + \lim_{M\to-\infty} \int_{-M}^0 x\ dx$ and show that at least one of these diverges (actually, they both do).

1c) Use PF's to get $\int_2^5 \frac{1}{x} + \frac{1/2}{x+1} + \frac{1/2}{x-1}\ dx = \ln|x|\ |_2^5 + \cdots$, etc. You can use "the shortcut" on this one to find the A, B and C but the standard way is also OK.

I generally gave less credit for other methods, which are harder and/or less reliable. For example, I am not quite sure that you *must* factor the $x^2 - 1$, but that is the standard method. Likewise, a few people tried $x = \sec\theta$, which probably ought to work, but didn't.

1d) This looks like it might be a partial fraction problem, but the denominator does not factor (if this is not clear, try the quadratic formula). So, complete the square, and then use a substitution such as $x+2=u$ or $x+2=\tan\theta$. Get $\int_{-3}^{-1}\frac{dx}{(x+2)^2+1} = \int_{-1}^{1}\frac{dx}{u^2+1} = \tan^{-1}u|_{-1}^{1} = \pi/2$.

1e) Geom, so $S=\frac{a}{1-r}=\frac{-10/3}{5/3}=-2$. Notice that a is the always the first term, regardless of whether it is from $k=0$ or $k=2$ or whatever (you can revise the series to start with $k=0$ if you like, but it is not worth the effort. If we needed s_n instead, more care would be required).

1f) D. Telescoping, so you can easily compute $s_n=(n+1)^3-1$, which diverges to $+\infty$. Or you could use the Divergence Test, etc, with some work and reasoning included. But if a series is special (telescoping, geometric, etc) you should usually take advantage of that.

1g) 63. The best safest way is the formula above with $n=3$, so 64-1=63. It is also OK to use $s_3=a_1+a_2+a_3=7+19+37=63$. But many people who used this second method made arithmetic mistakes.

2a) $x_2=1+2=3$ $x_3=3/2+2=7/2=3.5$.

2b) $x_{n+1}-x_n=2-x_n/2\geq 0$ because $x_n\leq 4$. So, the sequence is increasing. There are other approaches, but some lead to messy algebra. Be ready to try two or three.

2c) From above, it is monotone and bounded so it converges to some L. Take a limit of both sides of the recursion formula and get $L=L/2+2$, so $L/2=2$ and $L=4$. Some people were able to guess that $L=4$ but that is not reliable.

A few people read $x_{n+1}=x_n/2+2$ incorrectly as $x_{n+1}=x_n/4$. But we have a "precedence rule" that division comes before addition. This reduces the need for lots of parentheses. Also, why would anyone write $x_n/2+2$ for $x_n/4$?

3a) $3(\cdots)/6=11/2=5.5$. I suggest a quick sanity check on such problems. For example, the wall is roughly 3×2, so 5.5 seems plausible, but many exam answers were not. I generally give more partial credit for plausible answers.

3b) We cannot answer because we don't know whether f is concave up or down or neither. But I gave partial credit if you decided from a graph (maybe incorrectly) that f is concave up and that we have an overestimate.

4) The most common choice was the Comparison Test (with $b_k=2k/k^3=2/k^2$), which is OK if you include enough work, checking that $a_k\leq b_k$ and that $\sum b_k$ C. The Limit Comparison Test is also OK and maybe a little easier. But not the Divergence Test or Ratio Test. The Integral Test might work, but if so, it looks hard (see answer to the Bonus).

5) FTFFT

6) The Comparison Test was a little more popular, but some answers looked memorized,

with too many gaps and notational errors. The Divergence Test proof seems easier to me.

B) In hindsight a simple example like $\sum 1$ seems OK though the integral test fails on a technicality ($\lim f \neq 0$). Another example, maybe less clever, is $b_k = 2+\sin(k)$. The series diverges because $b_k \geq a_k = 1$. The oscillations prevent the Ratio and Integral tests (try them!).

If you prefer a convergent example, set $a_k = \frac{5+(-1)^k}{7k^2} \geq 0$, which is less than $b_k = \frac{1}{k^2}$. Again the oscillations prevent the other tests.

A few people gave examples similar to Problem 4 without much explanation and I did not give full credit. The Comparison Test works and the Ratio Test does not. So far, so good. But the Integral Test works, at least in theory. It might be hard to find an antiderivative of $\frac{2x}{x^3+1}$ (or etc) but if you think it is impossible you should explain that in your answer. In the case of Problem 4, partial fractions should produce an antiderivative.