Exam III | 2pm, Apr 5, 2018
MAC 2312 | S Hudson

1) [7 points each = 35 total] Compute and simplify (within reason). If it diverges, show some work and explain.

a) $\int_e^{+\infty} \frac{dx}{x\ln(x)}$

b) $\int_{-\infty}^{+\infty} \frac{4dx}{x^2+1}$

c) $\int \frac{dx}{x(x^2-1)}$

d) $\lim_{n\to\infty} \cos(n\pi)$

e) $\sum_{k=2}^{\infty}(2/3)^k$. Notice this starts at $k = 2$.

2) [10 pts] Use an infinite series to express the repeating $0.12\bar{3}$ in the form p/q. Notice the "bar" on the 3!

3) [20 pts] Classify each series as CA, CC or D. Justify by mentioning relevant test(s) with any needed calculations.

$\sum_{k=1}^{\infty}(\frac{2k+5}{3k+1})^k$

$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

$\sum_{k=1}^{\infty} \frac{-2}{7k-1}$

$\sum_{k=1}^{\infty} \frac{2^k}{k!}$

$\sum_{k=1}^{\infty} \frac{2+5(-1)^k}{k^2}$

4) [10 pts] Suppose the Midpoint Rule with $n = 10$ is used to estimate $\int_0^5 \sqrt{x}\, dx$.

4a) Will the estimate be greater than, equal to, or less than the integral ? Explain.

4b) Explain why we cannot bound the error using the usual formula, $|E_M| \le \frac{(b-a)^3 K_2}{24n^2}$.

5) (15pts) Answer True or False:

Every convergent sequence is bounded and monotonic.

The partial sums of the series $\sum_{k=1}^{\infty}(-1)^k$ are bounded.

Eventually the partial sums of the series $\sum_{k=1}^{\infty} 5 - k$ are monotonic.

We could use the Integral Test to show that $\int_2^{\infty} \frac{x^3}{x^4-1}\, dx$ diverges.

If $a_{n+1}/a_n \ge 3/4$ for all n, then $\{a_n\}$ is an increasing sequence.

6) (10 pts) Choose ONE proof, explain thoroughly:

a) State the main theorem about convergence of infinite geometric series. Prove it, including a calculation of s_n.

b) State and prove the Divergence Test.

BONUS: (5 pts) Give an example of a positive continuous function on $(0, \infty)$ such that $\int_0^\infty f(x)\, dx$ converges but $\int_0^\infty f(x)^2\, dx$ diverges. You do not have to completely prove your answer is correct, but explain your reasoning.

Remarks and Answers: The average was 56 based on the top 25 scores, which is a bit low. The high scores were 84 and two 81's. The average results on each problem were similar except problem 4 (about 40%), with slightly better results on 2 and 5. Here is an advisory scale for Exam 3:

A's = 69 to 100
B's = 57 to 68
C's = 47 to 56
D's - 37 to 46

Here is a rough scale for the semester, based on the three exams averaged:

A's = 74 to 100
B's = 64 to 73
C's = 54 to 63
D's = 44 to 53

1) *General remarks on problem 1 and similar problems.* As I mentioned before the exam, some answers are just not possible:

* The final answer will never be "Converges", and this will probably not even get partial credit. If it converges, you should normally compute a number (but not for 1c).

* You do not need any convergence tests, except maybe on a divergent series (not on integrals or sequences). And there are no divergent series in this problem. These remarks do not apply to Problem 3, of course, which has very different instructions, and is mostly about convergence tests.

* An indefinite integral like 1c cannot "diverge". Your answer should always look like "$F(x) + C$". An improper definite integral such as 1a or 1b can diverge - even without the "∞" as a clue.

1a) This improper integral diverges (to $+\infty$, but you did not have to say this). For full credit, your work should include $\lim_{M \to +\infty}$ near the start, and then $u = \ln(x)$.

1b) 4π. Also improper, this should be split into $\int_{-\infty}^0 f + \int_0^{+\infty} f$, and then handle the two parts with limits and $\tan^{-1}(x)$.

1c) $-\ln|x| + (\ln|x+1| + \ln|x-1|)/2 + C$. This can be simplified in various ways; for example, $\ln|x+1| + \ln|x-1| = \ln|x^2-1|$, but I didn't require that.

The standard method here is Partial Fractions. Factor $x^2 - 1$, solve for A,B and C (using the shortcut method if you like) and do the 3 easy integrals.

A few people started with $x = \sec\theta$, which ought to work, but it is harder, and did not usually end well. A few people used PFs, but failed to factor $x^2 - 1$, which didn't usually end well either. I am not sure this "mistake" is always fatal, but it is not reliable.

1d) Diverges. Notice that $\cos(\pi) = -1$ and $\cos(2\pi) = +1$, etc, so the a_k oscillate and don't converge.

Some people explained rather badly by saying that $\cos(x)$ (or just "cos") oscillates. But this same reasoning would imply that $\lim\sin(n\pi)$ diverges, which is not true. For these, it is better to write out some terms to see the pattern.

1e) $S = \frac{a}{1-r} = \frac{4/9}{1-2/3} = 4/3$. A common "mistake" here was to apply some convergence test and stop, without giving a numerical answer. Convergence tests are required in problem 3.

Another method from Ch.5 to deal with the k=2 is: 1+ 2/3 + ANS = $\sum_{k=0}^{\infty} ar^k = \frac{1}{1-2/3} = 3$. So, ANS = 3 - 5/3 = 4/3. This is a good method to know, but is not usually necessary with a geometric series.

2) Split it as $0.12 + 0.003 + 0.0003 + \cdots = 12/100 + \frac{a}{1-r}$ where $a = 0.003 = 3/1000$ and $r = 1/10$. This simplifies to 37/300. One clever student got this another way, from $1/3 - 0.21$.

3a) CA by the root test. In this answer and ones below I am leaving some calculations to you (such as $\rho = 2/3 < 1$).

3b) CC. It converges (by the AST) but not absolutely (it becomes a p-series with $p = 1/2 < 1$). You cannot apply many of our tests to the original series because most tests assume $a_k \geq 0$. See also the next problem.

3c) D. Notice the minus sign, which makes $a_k < 0$. So, you cannot apply the comparison test etc directly. You *can* factor out -2 and then study $-2\sum\frac{1}{7k-1}$ or simply $\sum\frac{1}{7k-1}$ instead. This series (with no minus sign) diverges by comparison with $\sum\frac{1}{7k}$, so the original series diverges too.

3d) CA by the Ratio Test ($\rho = 0$). With no negative terms, CA means the same as C, but the instructions ask for "CA".

3e) CA. This is a slightly tricky alternating series in which the a_k do not decrease monotonically. The simplest method is try for CA. $|\frac{2+5(-1)^k}{k^2}| \leq \frac{7}{k^2}$ (because $|2 \pm 5| \leq 7$). By comparison with the p-series, we conclude that $\sum|\frac{2+5(-1)^k}{k^2}|$ converges, so "CA".

You could split the series into $2\sum \frac{1}{k^2} + 5\sum \frac{(-1)^k}{k^2}$ and argue that a CA plus a CA makes a CA (true but not completely obvious). But if you got a D plus a D or some other combination, you'd be stuck (no valid conclusion), so a split is not always a great plan.

4a) Overestimate, because the function is concave down (see your lecture notes for more on this if necessary).

4b) K_2 does not exist because $f''(x) = -x^{-3/2}/4$ is not bounded for x near 0. You could just say $f''(0)$ doesn't exist (not as precise, but acceptable).

5) FTTTF

6) See the text. A common problem with the geometric one was to include the rather tricky calculation of s_n but to omit the main formula, $S = \frac{a}{1-r}$. Many people did not include a limit, or mention $|r| < 1$, or never stated the theorem at all.

Bonus) I don't have a familiar example, but with some thought, we can create one. As a first try, let $g(x) = x^{-1/2}$. Notice that $\int_0^1 g^2(x)\ dx$ diverges, but $\int_0^1 g(x)\ dx$ converges. Sadly $\int_1^\infty g(x)\ dx$ diverges, so we must revise the function. You can check that the related example below works (compute $\int_0^1 + \int_1^\infty$ for both f and f^2):

$$f(x) = \begin{cases} x^{-1/2} & \text{if } x \leq 1 \\ x^{-2} & \text{if } x \geq 1 \end{cases}$$