Given: $5! = 120$, $\sqrt{1/2} = .7$, $\sqrt[3]{1/3} = .4$. Also, I allowed people to use my basic calculator (but probably won’t repeat that).

1) [5pts] State the definition of $\ln(x)$ (which includes a definite integral).

2) [5pts] Find the volume, when the region under $y = \sin(x)$, above $y = 0$, and between $x = 0$ and $x = \pi$, is revolved around the $y$-axis.

3) [5pts] Find the general term, and the limit: $(\sqrt{2} - \sqrt{3})$, $(\sqrt{3} - \sqrt{4})$, $(\sqrt{4} - \sqrt{5}), \ldots$

4) [5pts each] Classify as D, CC or CA. Mention which test(s) you use, and show enough work to justify your answer [though you don’t have to explain as carefully as in a proof].

4a) $\sum_{k=1}^{\infty} (-1)^k \left( \frac{k+1}{2k+1} \right)^k$

4b) $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^{2}+1}$

5) [20pts] Answer True or False.

If $a + b > 2$ then $\sum_{k=1}^{\infty} \frac{1}{k^a+k^b}$ converges.

The polar equations $r = \sin 3\theta$ and $r = -\sin 3\theta$ have the same graphs.

If $u_k \rightarrow 0$ then $\sum_{k=1}^{\infty} u_k$ converges.

$f(x) = x^{1/3}$ has a MacLaurin series.

$\int \sin^3(x) \, dx = \frac{1}{3} \cos^3(x) - \cos(x) + C$

The average value of $\sin(3x)$ on $[0, 2\pi]$ is zero.

The alternating harmonic series converges conditionally.

If $0 < a_n < 1$ for all $n$, then $\{a_n\}$ converges.

The integral $\int_{1}^{+\infty} 1/x^p \, dx$ converges if and only if $p > 1$.

Simpson’s rule is exact if $f$ is a polynomial of degree 3.

6) [5pts each] Compute each integral that converges. Use limits if the integral is improper.

$$\int_{-\infty}^{+\infty} \frac{1}{1 + x^2} \, dx$$

$$\int_{1}^{e} \ln(x) \, dx$$
7) [7pts] Find the radius of convergence and the interval of convergence, for the series

\[ \sum_{k=1}^{\infty} \frac{x^k}{2^k(k+1)} \]

8) [10pts] a) Write out the McLaurin series for \( \cos(x) \), and then for \( \cos(x^2) \).

b) Write the definite integral \( \int_0^1 \cos(x^2) \, dx \) as an infinite series. For a little extra credit, include a formula for the general term.

c) Write out a specific partial sum which approximates the integral to 3 decimal places, and explain how you know it is correct. Don’t simplify the sum (unless you want to).

9) [10pts] a) Use \( T_4 \) (trapezoid rule) to approximate \( \int_0^1 \cos(x^2) \, dx \). You can leave the answer as a fraction, such as \( 7.777/17 \). You can use these [corrected] approximations, where the angle is given in radians:

\[
\begin{align*}
\cos(0) &= 1 \\
\cos(0.25^2) &= .998 \\
\cos(0.5^2) &= .969 \\
\cos(0.75^2) &= .846 \\
\cos(1) &= .540
\end{align*}
\]

b) Find an \( n \) so that \( T_n \) is within \( 10^{-4} \) of the integral above. You don’t have to compute \( T_n \). Your \( n \) should be the smallest possible (or pretty close) and your work should show that. Hint 1: \( |E_T| \leq \frac{K_2(b-a)^3}{12n^2} \). Hint 2: I used \( \sqrt{100}/6 \sim 4 \) and you can use similar rough estimates.

10) [10pts] Find the area of the region inside the cardioid \( r = 1 + \cos(\theta) \).

11) [8pts] Choose ONE to do (on the back);

A. State and prove the Comparison Test.

B. State and prove the first Fundamental Theorem of Calculus (about \( \int_a^b f \, dx \)).

C. State and prove the M-test (about bounded partial sums).

Remarks and Answers: The average was approx 57/100, with relatively low scores on most of the problems, especially 8 (integration of a power series) and 11 (the proof). It seemed that most people didn’t study for Chs 10-11 as thoroughly as the earlier material, maybe because of the time constraints of the short summer term. The rough scale is A’s = 71-100, B’s = 61-70, etc. I have not set the scale for the semester grades yet.

1) \( \ln(x) = \int_1^x \frac{1}{t} \, dt \), for \( x > 0 \).

2) Using shells and IBP, \( V = \int_0^\pi 2\pi x \sin(x) \, dx = 2\pi^2 \). [see HW 8.2.51]
3) \(a_k = \sqrt{k+1} - \sqrt{k+2}\) for \(k \geq 1\). Using a conjugate, \(a_k = \frac{-1}{\sqrt{k+1} + \sqrt{k+2}}\), which shows that the limit is zero. [A fancy alternative: note that \(a_k \sim -\frac{d}{dx}\sqrt{x}\)] Many people treated this as a telescoping series problem, but it’s about a sequence, not a series. [see 10.1.29]

4a) CA (Root test for AC; see the last lecture notes)

4b) CA (integral test, or comparison with the p-series, p=2) [see 10.4.19]

5) TFFFT TTFTT

6a) \(\pi\). Split in two and use limits, \(\tan^{-1}\).

6b) \(x \ln(x) - x|_1^1 = 1\) (use IBP).

7) RTAC: \(\rho = |x|/2\), so it converges on \((-2, 2)\), and \(R = 2\). It converges at one endpoint (AHS), but not the other (HS), so \(I = [-2, 2]\).

8a) \(\cos(x^2) = 1 - x^4/2! + x^8/4! - x^{12}/6! + \ldots\) Note: \((x^2)^4 = x^8\)

8b) Many people forgot to integrate, \(x - x^5/(5 \cdot 2!) + x^9/(9 \cdot 4!) - \ldots\) Or they forgot the \(\rho^0\), which gives \(1 - 1/(5 \cdot 2!) + 1/(9 \cdot 4!) - 1/(13 \cdot 6!)\ldots\) The general term is \(\frac{(-1)^k}{(4k+1)(2k)!}\).

8c) \(1 - 1/(5 \cdot 2!) + 1/(9 \cdot 4!)\). It is alternating (and the terms decrease to zero). Since \(13 \cdot 6! > 2000\), we can omit that term, and the rest. Note: It is quite hard to use the \(R_n(x)\) method here, mainly because \(M\) is so hard to estimate in this example (it’s similar to the \(K_2\) in 9b, but worse).

9a) 7.166/8. You should compare this with your answer to 8c) as a sanity check [you could round off 8c) to \(1.0 - 0.1 = 0.9\), for example].

9b) I got \(n = 70\) but your work is more important than the number. For, \(K_2\), get \(f'(x) = -2x\sin(x^2)\) and \(f''(x) = -4x^2\cos(x^2) - 2\sin(x^2)\). From the triangle inequality, the absolute value of this is at most 6, and you can set \(K_2 = 6\) (other methods might give other reasonable choices, but I wanted to see the work, and did not accept \(K_2 = 1\)). Then set \(\frac{6}{12\pi^2} \leq 10^{-4}\) and get \(n \geq 100\sqrt{1/2} = 70\) (approx), with help from page 1. The grades were fairly high on 9a, but low on 9b, a relatively hard 5-point problem.

10) \(\int_0^{2\pi} (1 + \cos(\theta)^2)/2 \, d\theta = 3\pi/2\).

11) See the text or lecture notes. Most people chose the Comparison Test, but many forgot to mention that the terms must be nonnegative, or forgot to include the \(\Sigma\)'s, or left out the proof.