MAC 2312 Dec 9, 2016

Final Exam Prof. S. Hudson

1) [10 pts] Classify each series as D for divergent, CA for absolutely convergent, or CC for conditionally convergent. Justify your answers with convergence tests and calculations.

$\sum_{k=2}^{+\infty} \frac{k(1+k^2)}{1+\sqrt{k}+5k^3}$

$\sum_{k=2}^{+\infty} (\frac{-2}{\ln k})^k$

2) [10 pts] Find the arc length of each curve, but just set them up. You can leave your answer as an integral.

a) $y = \sin(x)$ for $0 \le x \le \pi$.

b) $r = \sin(3\theta)$.

3) [10 pts] Compute each integral, or show it diverges. Remember to include a limit and/or a +C if needed.

$\int_0^\infty (1-x)e^{-x}\ dx$

$\int \tan^2 x \sec^4 x\ dx$

4) [10 pts] Compute the Riemann sum for $\int_0^\pi \sin x\ dx$ using the right endpoint rule with $n = 3$ and $x_1 = \pi/2$ and $\Delta x_2 = \pi/3$. You have enough info to compute x_2, Δx_1 and Δx_3, if you need those.

5) [10 pts] Find the first Taylor polynomial $p_1(x)$ for $f(x) = \ln(x)$ about $x_0 = 2$.

6) [20 pts] Answer True or False:

The McLaurin series for $\sqrt{1+x}$ converges for all real x.

$(n-3)/(n-\pi)$ is eventually decreasing.

$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$.

$e^{\ln(-2)} = -2$

If $|a_n| \le \frac{1}{n^2}$, then $\sum_{n=1}^{\infty} a_n$ converges.

A Riemann sum for $\int_0^4 \sin(x)\ dx$ must be positive.

S_{2n} is always between T_n and M_n, because it is a weighted average of them.

If a series converges conditionally, it can be rearranged to converge to 17.

Any two cardioids of the form $r = a + b\cos\theta$ and $r = c + d\sin\theta$ must intersect.

Distance traveled is always greater than displacement.

7) [10 pts] Use a series to approximate $\int_0^1 \frac{1-\cos x}{x}\, dx$ to three decimal place accuracy. You can leave your answer as a short sum of fractions. Discuss briefly why your answer is accurate enough. [Hints: 6!=720, 7!=5040, 8!=40320, 9!=362,880].

8) [10 pts] Sketch the curve $r = 1 + \sin\theta$, by plotting at least 4 points. Find the area of the region that is inside the curve and in the second quadrant (where $x < 0$ and $y > 0$).

9) [10 pts] Choose ONE proof:

a) Prove $\lim \frac{2^k}{k!} = 0$ using a series and convergence test(s).

b) State and prove the formula for area in polar coordinates. Include picture(s), trig, a sum, a limit, and enough explanation.

Remarks and Answers: The average was 52 / 100 with high scores of 89 and 74. The average was good on problem 1, but rather low (approx 40%) on 2, 4, 5 and 7. I have not set the scale for the semester yet.

1a) D, because $\lim a_k = 1/5$. Div Test.

1b) CA, using $|a_k|$ and the Root Test.

2a) $L = \int_0^\pi \sqrt{1 + (\cos\theta)^2}\, dx$

2b) $L = \int_0^\pi \sqrt{(\sin 3\theta)^2 + (3\cos 3\theta)^2}\, dx$. Do not use $\int_0^{2\pi}$ because of retracing. It is OK to use symmetry (use $3\int_0^{\pi/3}$, for example).

3a) $\lim_{M\to\infty} \int_0^M (1-x)e^{-x}\, dx = \lim_{M\to\infty} -(1-x)e^{-x}|_0^M - \int_0^M e^{-x}\, dx = 1 - 1 = 0$. This uses IBP, and ideally L'Hopital's Rule (though I allowed some educated guesswork on that step).

3b) $\tan^3(x)/3 + \tan^5(x)/5 + C$. This uses the standard trig sub, $u = \tan(x)$.

4) It is a good idea to draw a number line from 0 to π as a guide for the following calculations (which should be easy anyway). Deduce that $x_0 = 0$, $x_1 = \pi/2$, $x_2 = x_1 + \Delta x_2 = \pi/2 + \pi/3 = 5\pi/6$ and $x_3 = \pi$. Deduce that $\Delta x_1 = \pi/2$ and $\Delta x_3 = \pi/6$ (do not use $\Delta x = (b-a)/n$ since *this is not a regular partition*). Then plugging into the definition of a Riemann sum, with the RER, $\Sigma \sin(x_k)\Delta x_k = \sin(\pi/2)\pi/2 + \sin(5\pi/6)\pi/3 + \sin(\pi)\pi/6 = 2\pi/3$.

5) $p_1(x) = \ln(2) + (x-2)/2$ (using the standard formula for c_k with $k = 0, 1$).

6) FTTFT FTTTF

7) $1/4 - 1/96$. Here is an outline of the solution. There are other possible outlines, but if

you try to wander through this with *no plan* (from previous practice), you will probably get lost. Many people forgot to integrate, for example.

a) Replace the function by a power series, $f(x) = \cdots = x^1/2! - x^3/4! + x^5/6! + \cdots$.

b) Do the integration, and think about accuracy; $\int_0^1 f(x)\ dx = 1/(2 \cdot 2!) - 1/(4 \cdot 4!) + 1/(6 \cdot 6!) + \cdots \approx 1/4 - 1/96$ because this alternates and $1/(6 \cdot 6!) < .0005$.

8) You should recognize the formula as a cardioid. I think the graph is in the text; it has a cusp on the bottom.

$A = (1/2) \int_{\pi/2}^{\pi} (1 + \sin\theta)^2\ d\theta = \cdots = 3\pi/8\ + 1$.

A sanity check is advised. From the sketch, the region resembles a 2x1 rectangle, and the exact area turns out to be approx 2.

9a) Form a series from this and apply the Ratio Test to show it converges. Then apply the Divergence Test for the conclusion. This is in the text, and the lectures, as is 9b.