MAC 2312 May 5, 2016
Final Exam and Key Prof. S. Hudson

1) [8 pts] Find the volume of the solid generated when the region enclosed by $y = e^{-2x}$, $y = 0$, $x = 0$ and $x = 1$ is revolved around the x-axis.

2) [8 pts] Determine if the series converges. If so, find its sum.

$$\sum_{n=1}^{\infty} \frac{3}{5^n}$$

3) [8 pts] Compute this integral. Or, if it diverges, explain why:

$$\int_{-\infty}^{\infty} \frac{e^x \, dx}{1 + e^{2x}}$$

4) [6 pts] Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$

5) [8 pts] Find the radius and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{3^n \, n^{1/3}}$$

6) [6 pts] Given a series $\sum_{n=1}^{\infty} a_n$ with $a_7 = 7$, and with partial sums $s_6 = 60$ and $s_8 = 80$, find a_8.

7) [6 pts] Classify the series (as D, CA or CC) with a brief justification.
$\sum_{k=1}^{\infty} \frac{(-1)^k (k+2)}{k(k+3)}$

8) [8 pts] Use Maclaurin series to approximate $\int_0^1 \sin(x^2) \, dx$ to 2 decimal place accuracy. You can leave your answer as a fraction. Use as few terms of the series as possible and explain your method briefly (mainly, explain how you chose n).

9) [8 pts] Sketch the curve and find the area enclosed, $r = 3(1 + \sin\theta)$.

10) [5 pts each] Find the sum of each series by associating it with some known Maclaurin series (neither should require much calculation, these are more about memory). Your answer may include some common functions (for example, $3 + \tan^{-1}(2)$ would not have to be simplified).

$$1 - 1/2 + 1/3 - 1/4 + \cdots =$$

$$2 + 4/2! + 8/3! + 16/4! + \cdots =$$

11) [10 pts] Answer True or False.

The Maclaurin series for $\tan^{-1}(x)$ converges conditionally at $x = 1$ to $\pi/4$.

The average value of $\cos^2(4x)$ on $[0, 2\pi]$ is π.

If $|r| < 1$ then $r^2 + r^4 + r^6 + \ldots$ converges to $\frac{r^2}{1-r^2}$.

$\int_0^{3\pi/2} \sin(x)\ dx > \int_0^{\pi} \sin(x)\ dx$.

The graphs of $r = \sin 2\theta$ and $r = \sin 4\theta$ enclose equal areas.

12) [8 pts] Choose ONE to do (as usual, you can continue on the back, but leave a note);

A. State and prove the integral formula for area in polar coordinates.

B. State and prove the Comparison Test.

C. Prove that the Harmonic Series diverges, without convergence tests.

13) [6 pts] Use the Right Endpoint Rule to compute a Riemann sum for $\int_0^{3\pi/2} \sin(x)\ dx$ with $n = 3$, $\Delta_1 = \pi/2$, $\Delta_2 = \pi/3$ and $\Delta_3 = 2\pi/3$. So, for example, $x_2 = 5\pi/6$. Simplify completely and show all the work.

Bonus) [5 pts] Find a function $f(x)$ that has this Maclaurin Series:

$$x + 2x^2 + 3x^3 + \cdots$$

Remarks and Answers: The average was approx 50%, not including the very worst scores, with high scores of 89 and 80. The worst average was on problem 10 (26%) and the best was on problem 12 (71%). I do not make any special scale for the final exam.

1) The disk method is simplest. $V = \pi \int_0^1 (e^{-2x})^2 dx = -\frac{\pi}{4} e^{-4x}|_0^1 = \pi(1 - e^{-4})/4$.

2) It converges, because $r = 1/5 < 1$. Then $S = \frac{3/5}{1-1/5} = 3/4$.

3) $\pi/2$. Ideally, split this into two improper integrals (but they can be combined later) and then use $u = e^x$ to get $\int_0^{\infty} \frac{du}{1+u^2} = \ldots = \pi/2$ (since $\tan^{-1}(x) \to \pi/2$ at infinity). For full credit include the usual "lim".

4) After simplifying, $\ln|\sqrt{x^2+4} + x| + C$.

5) Use the RTAC to get $\rho = |x - 1|/3 < 1$, or $-2 < x < 4$. Then checking the endpoint, get $I = (-2, 4]$.

6) $a_8 = 13$, because $s_6 + a_7 + a_8 = s_8$.

7) CC. It converges by the AST (check briefly that a_k decreases to 0, or at least mention this). It does not CA because the new series can be compared to the Harmonic Series,

using the Limit Comp Test. For full credit, you needed to mention the AST and the LCT with some comments like these.

In principle, there are alternative explanations, but I don't recall any others working out. You cannot use the Comparison Test along with the $(-1)^k$. The RTAC does not help here because it leads to $\rho = 1$, which is inconclusive.

8) Since the results on this problem were fairly poor, this answer will be detailed, but slightly oversimplified. You need to get the series right and need to include the right number of terms. First, use

$$\sin(x) = x - x^3/3! + x^5/5! - \ldots$$

from memory. Sub in x^2 to get

$$\sin(x^2) = x^2 - x^6/3! + x^{10}/5! - \ldots$$

$$\int_0^1 \sin(x^2)dx = 1/3 - 1/42 + 1/(11 \cdot 5!) - \ldots$$

Phase 2 (when to stop?): Let $f(x) = \sin(x)$ (this seems simplest). Use $R_n \leq \frac{M}{(n+1)!}|x| \leq \frac{1}{(n+1)!} \leq .005 = 1/200$. This holds for $n \geq 5$, so let $n = 5$. This implies we can use the first three terms, and answer with $1/3 - 1/42 + 1/(11 \cdot 5!)$.

When grading, I did not care very much whether you chose n to be 4 or 5 or 6 (etc) but you had to provide some solid reasoning for your choice. The AST is a good alternative, probably simpler than R_n for this example, but I suggest R_n as a standard response.

9) $A = 27\pi/2$. See the text or lectures for similar graphs. This should be symmetric about the y-axis, with no retracing, so you can use $\int_0^{2\pi}$ on it, or other methods.

10a) This is the famous A.H.S., which converges to $\ln(2)$ as mentioned several times in class. It is also the series for $\ln(x+1)$ at $x = 1$.

10b) $e^2 - 1$. Write out the famous series for e^x and set $x = 2$, and you get the given series, with an extra $+1$. These two problems came from the Ch.9 Review.

11) TFTFT. The last one may be a little surprising, but you can check that the number of petals does not affect the total area as long as it is an even number.

12) See the text. Most people chose the Comparison Test and did OK. Common mistakes:

i) Write most of the proof, but never actually *state* the theorem. In most answers, the proof was not clear enough that a reader could infer the statement easily.

ii) Forgetting to assume $a_k \leq b_k$, or inserting extra hypotheses, such as 'a_k is monotone'. Some people treated the two series as equivalent, which is even worse (eg, 'if one converges, the other converges'. This phrase might fit other tests, but not the CT).

iii) Ideally, you should mention that the *partial sums* of both series are monotone, and also discuss whether each is a bounded sequence. But I let some minor gaps go if you included most of the logic.

13) 0. The x_i are $\pi/2$, $5\pi/6$ and $3\pi/2$. These are the 'partial sums' of the Δx's. The y_i are $\sin(\pi/2) = 1$, $\sin(5\pi/6) = 1/2$ and $\sin(3\pi/2) = -1$. These are the heights of the 3 rectangles. The R.Sum is $\sum y_i \Delta x_i = (1)(\pi/2) + (1/2)(\pi/3) + (-1)(2\pi/3) = 0$.

This is similar to a problem from Exam 1 that had poor results, and to old HW problems.

B) This kind of problem can be very hard, but this example has a fairly clear pattern. It is

$$x(1 + 2x + 3x^2 + \cdots) = x\frac{d}{dx}(1 + x + x^2 + \cdots) = x\frac{d}{dx}\frac{1}{1-x} = \frac{x}{(1-x)^2}$$

So, $f(x) = \frac{x}{(1-x)^2}$. Three people got this, or got close enough for partial credit.