MAC 2312 — June 24, 2016

Final Exam — Prof. S. Hudson

1) [15 pts] Compute each, showing all work:

$$\int x^3 \ln(x)\ dx$$

$$\int_0^{\pi/4} \tan^3 t \sec^3 t\ dt$$

$$\int_0^{\infty} te^{-t^2}\ dt$$

2) [10 pts] Find an integral (set-up only) for the volume of the solid obtained by revolving the region bounded by $x = 0$, $x + y = 6$ and $y = 0$ around the line $x = -1$. Shells are suggested, but other methods are OK. For 2 extra points, compute the integral.

3) [20 pts] Classify each series (as D, CA or CC). State the tests you are using, with brief justifications. Consider the limit comparison test and the root test.

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{3/4}}$

$\sum_{k=1}^{\infty} \frac{(-4)^k}{k^2}$

$\sum_{k=1}^{\infty} (\frac{2k-1}{3k+2})^k$

$\sum_{k=1}^{\infty} \frac{4k^2-2k+6}{8k^7+k-8}$

4) [5 pts] Find the first 3 nonzero terms of the Maclaurin series for $f(x) = \sin^2(x)$ using a trig identity and a known Maclaurin series.

5) [7 pts] Find the radius and interval of convergence of the series and, as usual, decide whether the endpoints belong. $\sum_{k=1}^{\infty} \frac{(-1)^k (x-1)^k}{k}$.

6) [10 pts] Sketch the curve and find the area enclosed, $r = 3\sin(3\theta)$.

7) [8 pts] Approximate $\sqrt{e}$ to 3 decimal places. You can leave a few fractions in your answer. Discuss briefly how you got the desired accuracy. You may use any of these, if needed: $e \approx 2.7$, $\sqrt{e} \approx 1.5$, $6! = 720$, $9! = 362{,}880$, $2^7 = 128$.

8) [15 pts] Answer True or False.

The Maclaurin series for $\frac{1}{1+x^2}$ converges for all x.

The graphs of $r = \sin 3\theta$ and $r = -\cos 3\theta$ are the same.

$\int_3^0 \sin(x)\ dx = \int_0^3 \sin(-x)\ dx$.

If $0 \leq a_n \leq b_n$ and $\lim b_n$ converges, then $\lim a_n$ converges.

The Maclaurin series for $\cos(x^3)$ converges for all real x.

9) [10 pts] Choose ONE to do.

A. State and prove the integral formula for area in polar coordinates.

B. State and prove the Divergence Test.

C. Show that $\sum_{k=1}^{\infty} k^{-2}$ converges using an integral, a picture and enough words (as done in class). This is meant to partially justify the Integral Test. Do not USE this test, or the p-series test, in your answer.

Bonus) [5 pts] Let $x_1 = 1$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for $n \geq 1$. Given that this recursively defined sequence converges to some number L, find a simple formula for L. Then approximate L using x_3.

Remarks and Answers: The average score (among the top 70%) was 56 out of 100, with high scores of 89 and 84. The average results were OK on most of the problems except for problem 4 (31%) and the True-False (47%). I do not set a scale for the final and have not yet revised the scale for the semester.

1a) $\frac{x^4 \ln(x)}{4} - \frac{x^4}{16} + C$

1b) Use $u = \sec(t)$. $\int_1^{\sqrt{2}} u^4 - u^2\ du =$ etc.

1c) 1/2. Use a substitution, $u = t^2$ or $u = -t^2$, and a limit, since this is improper. Avoid notation such as $F(t)|_0^{\infty}$, at least on exams.

2) Using shells, $V = 2\pi \int_0^6 (x+1)(6-x)\ dx = \ldots = 108\pi$ (using $r = x+1$). Using washers, $V = \pi \int_0^6 (7-y)^2 - 1^2\ dy$ (using $R = x+1 = 7-y$). You should communicate your method by writing 'Shells', drawing a picture and adopting the usual dx or dy, though all this was a very minor part of the grading.

3) Generally the correct conclusion was worth 3 points each on 3abcd, with no partial credit (except perhaps for C instead of CA). If that part was correct, a valid justification was worth 2 more points. The convergence tests mentioned on this Key are the only ones that worked out well.

3a) CC, by the AST and p-series.

3b) D, by the DT or the RTAC.

3c) CA by the Root Test ($\rho = 2/3$). Since $a_k \geq 0$, C is essentially the same as CA, but the instructions ask for 'CA'.

3d) CA, by the LCT, with $b_k = k^{-5}$.

4) $\sin^2(x) = 1 - \cos(2x))/2 = \ldots = x^2 - x^4/3 + 32x^6/6! - \ldots$ I also gave credit to one person who multiplied the series for $\sin(x)$ by itself, though that did not exactly follow the instructions.

5) $I = (0, 2]$, mainly from the RTAC, with $R = 1$ (half the length of I). There were many mistakes checking the endpoints, and many people did not answer about R.

6) $9\pi/4$. I suggest 3 times the area of one petal, $A = (3/2)\int_0^{\pi/3}(3\sin 3\theta)^2\, d\theta$, but $A = (1/2)(1/2)\int_0^{2\pi}(3\sin 3\theta)^2\, d\theta$ is also OK.

A few people used $A = (1/2)\int_0^{\pi}(3\sin 3\theta)^2\, d\theta$, which is technically OK, but the reasoning seems slightly bizarre (or perhaps this was just lucky). This plan should be explained for full credit.

7) $\sqrt{e} = e^{1/2} = 1 + 1/2 + 1/8 + \cdots + (1/2)^5/5!$, from the usual series for e^x with $x = 1/2$, stopping at $n = 5$. For the accuracy part, which many people omitted, $M = e^{1/2} \approx 1.5$, so

$$|R_5(1/2)| \leq \frac{1.5|1/2 - 0|^6}{6!} < 3/10,000 < 0.0005$$

using the data provided, but this inequality fails for $n < 5$. In my opinion, it is not very important whether you got $n = 5$ exactly, as long as you used the standard $|R_n(1/2)|$ method, mostly correctly. It is *not* OK to set $a_{n+1} < 0.0005$ instead, since this is not an alternating series.

8) FFTFT. Almost everyone missed the fourth one, perhaps confusing it with the Comparison test (which is about *series*, not sequences). Counter-example: $a_n = |\sin(n)|$ and $b_n = 1$. In effect, part 4 says a bounded sequence must converge, which is false.

This problem appears on one of my old exams. I do not suggest memorizing old TF examples, but do suggest TF practice, as needed. You will learn to read more carefully ($\lim a_n$, not $\sum a_n$) and sceptically (this resembles the CT but it is not the CT) and to use specific examples as guides.

9) (see the text or lectures). Most people chose B, usually with better results than on A and C. Those who chose A generally failed to include a Riemann sum. Those who chose C generally failed to mention partial sums - pretty serious problems.

Bonus) $L = \sqrt{2}$ and $x_3 = 17/12 \approx 1.42$.