Final Exam, Key | Dec 10, 2020
MAC 2312 | S Hudson

1) [5] Find the rectangular coordinates of the point $(\sqrt{2}, \pi/4)$, given in polar coordinates.

2) [7] Compute $\int x^3 \ln x \; dx$

3) [7] Compute $\int \frac{1}{x^2-7x+6} \; dx$

4) [7 each] Classify each series. You can answer with CC, CA or D, but justify your answers, usually with a convergence test and a calculation.

a) $\sum_{k=1}^{\infty} (-1)^k \frac{4}{3k-1}$

b) $\sum_{k=0}^{\infty} \frac{(-3)^k}{k!}$

5) [7] Let $r = \cos(3\theta)$ (a rose). Find the area enclosed by this curve.

6) [7] Again, let $r = \cos(3\theta)$. Find dy/dx at the point where $\theta = \pi/4$.

7) [10] Find the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{x^k}{k+1}$.

8) [7] A conical water tank is 9 ft tall with a radius of 3 ft at the top. It contains water to a depth of 6 ft with density 62 lbs per cubic ft. How much work is required to pump the water over the top of the tank ? Express your answer as a definite integral, with your reasoning. You do not have to compute it.

9) [7] Find the Taylor series for $f(x) = e^x$ at the point $x_0 = 2$. For maximum credit, use summation notation.

10) [7] Let R be the region bounded by $x = 0$, $y = x^{1/3}$ and $y = 3$. Find the volume of the solid obtained by revolving R around the y-axis. For maximum credit, sketch the solid and describe your method (disks, washers, shell, etc). Evaluate the integral, but you do not have to completely simplify the number.

11) [15] Answer True or False:

The function $f(x) = 1/x^2$ is integrable on $[-3, 3]$.

The function $f(x) = \tan^{-1}(x)$ is integrable on $[0, 2]$.

The average value of $f(x) = 5 + \sin x$ on an interval $[a, b]$ is always at least 5.

Every Riemann sum for $\int_0^3 2x \; dx$ is less than 10.

There are more than two points on the cardioid $r = 1 + \cos\theta$ with a vertical tangent line.

12) [7] Choose ONE proof, explain thoroughly:

a) State the formula for Area using an integral in polar coordinates and justify it with pictures, a Riemann sum and a clear explanation.

b) State and prove the Divergence Test.

Bonus [5]: Compute π to two decimal place accuracy as a finite sum of fractions and explain your reasoning. You do not have to simplify the sum.

Remarks, Answers: The average grade was approx 64, which is not bad for a final exam. There were 3 scores in the low 80's. This exam will probably not affect the scale much, but if the HW grades are high it might go up a couple of points. The grades were relatively high on problem 3 (80%) and low on problems 6, 7 and 8 (43%).

1) (1,1).

2) $\frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$.

3) $\frac{\ln|x-6|}{5} - \frac{\ln|x-1|}{5} + C$.

4a) CC, using the AST for C, and the CompT for "not CA".

4b) CA, using RTAC.

5) Sketch the rose carefully and notice the retracing. Note that $\cos 3\theta = 0$ when $\theta = \pi/6$.

The area of the top half of one petal is $A = (1/2)\int_0^{\pi/6} \cos^2 3\theta \, d\theta = \cdots = \pi/24$. There are three petals, so the final answer is $A = 6\pi/24 = \pi/4$.

Another approach is $A = (1/2)\int_0^{2\pi} \cos^2 3\theta \, d\theta = \cdots = \pi/2$. But this ignores retracing and is 2x too big. So, the final answer is $A = (\pi/2)/2 = \pi/4$.

The usual shortcut, replacing the integrand $\cos^2 3\theta$ by 1/2, gives the correct answer both times. But unless you are sure that works, using the trig identity is better.

6) 2. Start with $\frac{dy}{d\theta}/\frac{dx}{d\theta} = \cdots$, and $y = \cos(3\theta)\sin\theta$ etc. It gets a bit messy but simplifies to 2. People usually did OK on this one if they knew the first step.

7) $I = [-1, 1)$. The RTAC shows that I contains (-1,1) and maybe also 1-2 endpoints, which you must check separately.

8) $62\pi \int_0^6 (y/4)^2(12-y)\, dy$ (and stop). There were a couple of special exams with a 9 foot tank, with a slightly different answer. People who worked with a Riemann sum to reason out the formula generally did better than people who relied mainly on memory.

9) $\sum_{k=0}^{\infty} \frac{e^2(x-2)^k}{k!}$. Every $f^{(k)}(2) = e^2$, so the coefficients have a simple pattern (but several people thought $f^{(k)} = e^0$ or e^x, etc).

10) Using Disks (based on a horizontal line segment), $V = \int_0^2 \pi (y^3)^2 \, dy = \cdots = \frac{2^7\pi}{7}$.
It is OK to use Shells, but that is maybe a little harder. Not washers.

11) FTFFT

12) See the text.

B) Start from $\pi/4 = \tan^{-1}(1) = 1 - 1/3 + 1/5 \cdots$ (from a class on Taylor Series) and decide when to stop using either the R_n method the AST method. Since the instructions say "Compute", I gave little credit for answers like 314/1000, based mainly on memorization.