12.1-12.4 HOMEWORK

Graded by Learning Assistant Enzo De Oliveira.

I numbered the problems from each section and used a random number generator to select one problem from each section for 4 problems total. I then listed all the problems and selected a random problem.

The questions graded were

- Section 12.1: 22
- Section 12.2: 25
- Section 12.3: 15
- Section 12.4: 7, 29

Each question was out of 16 points, for a total of 80 points for accuracy. Another 20 points was awarded for neatness, legibility, work, stapling etc.

1. Section 12.1: 22

Describe the sets of points in space whose coordinates satisfy the given inequalities or combinations of equations and inequalities.

a) $x = y, z = 0$
b) $x = y$, no restriction z

Solutions:

a) The line $x = y$ on the x-y plane.
b) The plane that includes the line $y = x$ on the x-y plane and has an orthogonal normal vector to the z-axis ($\vec{k}$); parallel to the z axis.

Part A was out of 8 points. Key words were line, x-y plane, 45 degrees.

Part B was out of 8 points. Key words were plane, preposition: e.g. parallel to.

2. Section 12.2: 25

Express the vector as a product of its length and direction.

$$\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$$

Solution:

$$||\vec{v}|| = \sqrt{2^2 + 1^2 + 9 - 2)^2} = \sqrt{9} = 3$$

$$\vec{v} = 3(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k})$$

Finding the magnitude was 8 points. Showing the vector as a product of its length and direction were another 8 (The length was 4 points and the direction was 4 points.)

3. Section 12.3: 15

The direction angles α, β, and γ of a vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ are defined as follows:

- α is the angle between $\vec{v}$ and the positive x-axis $(0 \leq \alpha \leq \pi)$
- β is the angle between $\vec{v}$ and the positive y-axis $(0 \leq \beta \leq \pi)$
- γ is the angle between $\vec{v}$ and the positive z-axis $(0 \leq \gamma \leq \pi)$

a) Show that

$$\cos\alpha = \frac{a}{||\vec{v}||}, \cos\beta = \frac{b}{||\vec{v}||}, \cos\gamma = \frac{c}{||\vec{v}||}$$

and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the direction cosines.

b) Unit vectors are built from the direction cosines. Show that if $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ is a unit vector, then a, b, and c are the direction cosines of $\vec{v}$.

$$\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$$

Solution:

Part A

$$\cos\alpha = \frac{a\cdot 1 + b\cdot 0 + c\cdot 0}{||\vec{v}||\cdot||\vec{i}||} = \frac{a}{||\vec{v}||}$$

$$\cos\beta = \frac{a\cdot 0 + b\cdot 1 + c\cdot 0}{||\vec{v}||\cdot||\vec{j}||} = \frac{b}{||\vec{v}||}$$

$$\cos\gamma = \frac{a\cdot 0 + b\cdot 0 + c\cdot 1}{||\vec{v}||\cdot||\vec{k}||} = \frac{c}{||\vec{v}||}$$

Each of the above were worth 2 points.

Next,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$\left(\frac{a}{||\vec{v}||}\right)^2 + \left(\frac{b}{||\vec{v}||}\right)^2 + \left(\frac{c}{||\vec{v}||}\right)^2$$

$$\frac{a^2 + b^2 + c^2}{||\vec{v}||^2}$$

$$\frac{||\vec{v}||^2}{||\vec{v}||^2} = 1$$

This mini-proof was worth 4 points. Key point was to substitute and show that $a^2 + b^2 + c^2 = ||\vec{v}||^2$.

Part B If $\vec{v}$ is a unit vector then the magnitude of $\vec{v} = 1$. So

$$\cos\alpha = \frac{a}{||\vec{v}||} = a$$
$$\cos\beta = \frac{b}{||\vec{v}||} = b$$
$$\cos\gamma = \frac{c}{||\vec{v}||} = c$$

This part was worth 6 points. Had to explain why $||\vec{v}|| = 1$.

4. Section 12.4: 7

Find the length and direction of $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$

$$\vec{u} = -8\vec{i} - 2\vec{j} - 4\vec{k}, \vec{v} = 2\vec{i} + 2\vec{j} + \vec{k}$$

Solution:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix} = (-2+8)\vec{i} - (-8+8)\vec{j} + (-16+4)\vec{k} = 6\vec{i} + 0\vec{j} - 12\vec{k} = 6\vec{i} - 12\vec{k}$$

$$\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v}) = -(6\vec{i} - 12\vec{k}) = -6\vec{i} + 12\vec{k}$$

4 points were awarded for each correctly calculated cross product.

$$||\vec{u} \times \vec{v}|| = ||\vec{v} \times \vec{u}|| = \sqrt{6^2 + 12^2} = \sqrt{180} = 6\sqrt{5}$$

4 points were awarded for correctly calculating the magnitude.

$$\frac{\vec{u} \times \vec{v}}{||\vec{u} \times \vec{v}||} = \frac{6\vec{i} - 12\vec{k}}{6\sqrt{5}} = \frac{1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{k}$$
$$\frac{\vec{v} \times \vec{u}}{||\vec{v} \times \vec{u}||} = \frac{-6\vec{i} + 12\vec{k}}{6\sqrt{5}} = \frac{-1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{k}$$

2 points were awarded for showing the direction of each vector above.

5. Section 12.4: 29

Given nonzero vector $\vec{u}$, $\vec{v}$, and $\vec{w}$, use dot product and cross product notation, as appropriate to describe the following.

a) The vector projection of $\vec{u}$ onto $\vec{v}$
b) The vector orthogonal to $\vec{u}$ and $\vec{v}$
c) A vector orthogonal to $\vec{u} \times \vec{v}$ and $\vec{w}$
d) The volume of the parallelepiped determined by $\vec{u}$,$\vec{v}$,and $\vec{w}$
e) A vector orthogonal to $\vec{u} \times \vec{v}$ and $\vec{u} \times \vec{w}$
f) A vector of length $||\vec{u}||$ in the direction of $\vec{v}$

Solutions:

a) $proj_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2} \cdot \vec{v}$ or $\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} \cdot \vec{v}$

b) $\vec{u} \times \vec{v}$ or $\vec{v} \times \vec{u}$
c) $(\vec{u} \times \vec{v}) \times \vec{w}$ or $\vec{w} \times (\vec{u} \times \vec{v})$
d) $(\vec{u} \times \vec{v}) \cdot \vec{w}$ or $(\vec{v} \times \vec{w}) \cdot \vec{u}$ or $(\vec{u} \times \vec{w}) \cdot \vec{v}$
e) $(\vec{u} \times \vec{v}) \times (\vec{u} \times \vec{w})$
f) $||\vec{u}|| \cdot \frac{\vec{v}}{||\vec{v}||}$

If you answered

- 0-1 items correctly, you earned 0 points
- 2-3 items correctly, you earned 6 points
- 4-5 items correctly, you earned 12 points
- 6 items correctly, you earned all 16 points

Common Mistakes and feedback:

- Read the directions to make sure you are doing all the parts.
- Skipped steps.
- Changing vector notation led to mistakes. Stick with one. Emphasize those pointy brackets.
- Stop using Chegg. Some of you had very similarly worded answers.
- Staple your work so you do not lose any papers.

If you believe there were any mistakes, or have questions about how I graded your work, feel free to email me or show up to the LA sessions.