MAC 2313 Sept 25, 2017
Exam I and Key Prof. S. Hudson

1) [10 pts] Find the center and radius of the sphere that has $(2, 1, 3)$ and $(6, 5, 3)$ as endpoints of a diameter.

2) [10 pts] Find the *acute* angle formed by two diagonals of a cube. A diagonal connects two opposite corners and goes through the center. If you get an answer like $\theta = 2\sin^{-1}(1/5)$, for example, you do not have to simplify.

3) [10 pts] Where does the line $x = 3 + 2t, y = 3 - 2t$ intersect the parabola $y = x^2$? Answer with x and y coordinates, of course.

4) [10 pts] Find the equation of the plane through $(1, 2, 5)$ that is parallel to the plane $3x + 2y - z = 17$.

5) [10 pts] The formula $4x^2 + y^2 + z^2 = 4$ defines a quadric surface. Its intersection with $y = 1$ is a trace, and is a conic section. Find the equation of this trace and identify what kind of conic section it is (eg a hyperbola, or whatever).

6) [10 pts] Find the spherical coordinates of New Orleans as done in class (and in Ex.11.8.4). Assume the earth has radius 4000 miles, and that the x-axis goes though both the equator and the prime meridian (which also contains Greenwich, England). New Orleans is located at 90° west longitude and 30° north latitude.

7) [10 pts] Sketch a graph of $\cos t\mathbf{i} + \sin t\mathbf{j} + 2\mathbf{k}$, with $0 \le t \le 2\pi$. Describe the shape briefly in words.

8) [5 pts] Let $\mathbf{v} = \langle 3, 2, 1\rangle$ and $\mathbf{w} = \langle 2, 3, 4\rangle$. Compute $\mathbf{v} \times \mathbf{w}$ and check that $\mathbf{v} \perp \mathbf{v} \times \mathbf{w}$.

9) [15 pts total] Answer True or False. Assume $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are arbitrary vectors in R^3.

Every ellipsoid is a surface of revolution.

If two planes in R^3 with normal vectors $\mathbf{N_1}$ and $\mathbf{N_2}$ do not intersect then $\mathbf{N_1} \times \mathbf{N_2} = \mathbf{0}$.

If $P(x_0, y_0, z_0)$ lies in the xy-plane and in the yz-plane, then $x_0 = z_0 = 0$.

$\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

The natural domain of a vector-valued function is the intersection of the domains of its component functions.

10) [10 pts] Choose ONE and prove it. A good proof will probably be mostly words, along with some calculations, but you can also include pictures. You can answer on the back.

a) State and prove a formula for z in spherical coordinates.

b) The plane in R^3 through $P(x_0, y_0, z_0)$ with normal vector $\mathbf{N} =< n_1, n_2, n_3 >$, has the equation $n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$.

c) Thm 11.4.6a, that $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. If you like, you can assume $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) > 0$ for simplicity.

Remarks, Scale, Answers: The average was 66 / 100 based on the top 22 scores which is fairly normal. The two high scores were 98 and 97. The results on all the problems were pretty similar except for problems 6 (40%) and 8 (95%). Here is an advisory scale for Exam I:

A's 75 to 100
B's 65 to 74
C's 55 to 64
D's 45 to 54

1) Center = $(4, 4, 3)$ (by averaging coordinates). Radius = distance from center to either point = $\sqrt{8}$. It is also OK to compute and use $\overrightarrow{PQ} = \langle 4, 4, 0\rangle$ to get these answers, but that is just a bit longer.

2) $\theta = \cos^{-1}(1/3)$. Since the size and location of the cube are not specified, and not important, you can choose the corners. For example, let the center be $(0, 0, 0)$ with the eight corners at $(\pm 1, \pm 1, \pm 1)$. It is best to be specific like this, rather than (x, y, z) or whatever. Two of the diagonals (for example) have direction vectors $\mathbf{v} = \langle 1, 1, 1\rangle$ and $\mathbf{w} = \langle 1, 1, -1\rangle$. Taking the dot product, we get $\cos\theta = \frac{1}{\sqrt{3}\sqrt{3}}$.

If you choose $\mathbf{w} = \langle 1, -1, -1\rangle$ instead, maybe a slight mistake, you will get a negative dot product. This indicates an obtuse angle. You can correct the result by taking the supplementary angle, $\theta = \pi - \cos^{-1}(-1/3)$, which is the same as $\cos^{-1}(1/3)$.

3) Set $3 - 2t = (3 + 2t)^2$, solve, and get $t = -3$ or $t = -1/2$. So, there are two points, $(-3, 9)$ and $(2, 4)$.

4) $3(x - 1) + 2(y - 2) - (z - 5) = 0$. Not much work is required for this one, but that can help with partial credit if there is some silly mistake. Common errors were to forget the "$= 0$" or to use "$= 17$".

5) It is an ellipse, $4x^2 + z^2 = 3$.

6) $(4000, 3\pi/2, \pi/3)$. See the text or lecture notes.

7) It is a horizontal circle of radius 1 centered at (0,0,2).

8) $\langle 5, -10, 5\rangle$ and, to check, the dot product is $15 - 20 + 5 = 0$.

9) FTTFT

10) See the text or lecture notes for the proofs. For each of these, I'd suggest a guiding picture, though that is not strictly required. About 10c, for example, it seemed that about half the people who chose it had studied for it. Among those, the proofs were generally on the right track, but there were many notational errors. For example, Area $= \mathbf{v} \times \mathbf{w}$" (a vector??). This should be Area $= ||\mathbf{v} \times \mathbf{w}||$. Sometimes such errors accumulated until the proof was unreadable.

If interested, here is a list of exercises I looked at in writing this exam, though some were not chosen and some were modified. The first two are Examples worked out in the text - 12.1.E6b, 11.8.E4, 11.7.9a, 11.6.33, 11.5.17, 11.4.19, 11.3.14, 11.3.22, 11.2.31, 11.1.9, 11.1.17.