MAC 2313 Jan 26, 2017
Exam I Prof. S. Hudson

1) [10 pts] Find two unit vectors in R^2 parallel to the line $y = 3x + 1$.

2) [10 pts] Sketch a graph of $y = \sin(x)$ in 3-space. Include several traces, such as where $z = 1$ and $x = \pi$.

3) [10 pts] Find the distance from the point $P(1, 1) \in R^2$ to the line $y = 2x+1$. Hint: find a normal vector to the line, and then a projection.

4) [10 pts] Find the area of the triangle with vertices $P(1, 3, 5)$, $Q(0, 0, 0)$ and $R(3, 2, 1)$.

5) [10 pts] Find parametric equations for the line through the two points $P_1(2, 3, 4)$ and $P_2(5, 0, 7)$. For maximal credit, write the answer old style, not as a vector equation.

6) [10 pts] Find two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^3 such that
$\mathbf{u} + 2\mathbf{v} = 3\mathbf{i} - \mathbf{k}$ and
$3\mathbf{u} - \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

7) [10 pts] Find the equation of the plane whose points are equidistant from $P(2, -1, 1)$ and $Q(2, 1, 7)$. For example, $R(1, -3, 5)$ is in this plane because $||\overrightarrow{PR}|| = ||\overrightarrow{QR}|| = \sqrt{21}$. Hint: $\overrightarrow{PQ}$ is perpendicular to the plane.

8) [20 pts] Answer True or False. Assume $\mathbf{v}$ (etc) are arbitrary vectors in R^3 unless stated otherwise.

If $\mathbf{u} \perp \mathbf{v}$ and both are unit vectors, then $(3\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + 2\mathbf{v}) = 5$.

If $||\mathbf{v}|| = ||\mathbf{v} + \mathbf{w}||$ then $\mathbf{w} = \mathbf{0}$.

There are exactly two unit vectors that are parallel to any given nonzero vector.

If $P(x_0, y_0, z_0)$ lies in the xy-plane and in the yz-plane, then $x_0 = 0$.

$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

$\text{proj}_{2\mathbf{u}}\mathbf{w} = \text{proj}_{3\mathbf{u}}\mathbf{w}$

If $\text{proj}_{\mathbf{u}}\mathbf{w} = \mathbf{w}$ then $\mathbf{w} = \mathbf{u}$.

9) [10 pts] Choose ONE and prove it. You can answer on the back, but if so, leave a note here.

a) State and prove the formula for $\mathbf{u} \cdot \mathbf{v}$ (the one with θ) in R^2.

b) The plane in R^3 through $P(x_0, y_0, z_0)$ with normal vector $\mathbf{N} =< n_1, n_2, n_3 >$, has the equation

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

c) The volume of the parallelpiped that has $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ as adjacent edges is $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$.

Remarks and Answers: The average among the top 23 scores was 73 out of 100, which is good. The top two scores were 97 and 86, with five more scores in the 80s. The lowest scores were on problem 3 (48%) and the TF (60%), with the highest on problems 4, 5 and 9 (approx 90% each). The advisory scale is:

A's 81 to 100
B's 71 to 80
C's 61 to 70
D's 51 to 60

1) $\mathbf{u} = \frac{1}{\sqrt{10}}\langle 1, 3\rangle$ and $\mathbf{v} = -\mathbf{u} = \frac{1}{\sqrt{10}}\langle -1, -3\rangle$. These are the only two correct answers, but other notation for them is OK, such as $\mathbf{v} = \frac{-2}{\sqrt{40}}\mathbf{i} - \frac{6}{\sqrt{40}}\mathbf{j}$.

You can find the direction of the line, as a vector, by finding any two points on the line, and subtracting to get $\overrightarrow{PQ} = \langle 1, 3\rangle$, or $\langle 2, 6\rangle$, for example.

2) See me or your LA or software. The surface looks like a wavy vertical wall staying near the xz-plane.

When grading this, I tried not to count artistic skills very heavily, but your sketch should look like a 2D surface in 3-space. Your z-axis should point upwards, as usual. In principle, it should extend infinitely far in various directions, but I did not count that. The trace where $z = 0$ is a wavy line though the origin, staying the in the horizontal xy-plane. The trace where $x = \pi$ (or $x =$ any constant) is a vertical line. Traces like that one are easy to draw, so I'd suggest including many of those.

3) $d = \frac{2}{\sqrt{5}}$. This is the norm of the vector projection of $\mathbf{v}$ on $\mathbf{n}$ (which is also the absolute value of the scalar projection). Here, $\mathbf{v}$ is a vector from $P(1, 1)$ to any point on the line, such as $Q(0, 1)$. So, $\mathbf{v} = \overrightarrow{PQ} = \langle -1, 0\rangle$ works. You can get the normal vector $\mathbf{n} = \langle 2, -1\rangle$ from the equation of the line $2x - y = -1$, the same way as for a plane. A scalar multiple of $\langle 2, -1\rangle$ is OK. Some people got $\mathbf{n}$ from a careful graph and a little geometry (rise over run, etc).

General advice:

a) If you don't know how to start on a problem, a picture may remind you of a plan you have studied, or even suggest a new plan. In this example, your picture should include (at least) the line, a normal vector and points P and Q. Hopefully, it suggests using a projection, and maybe the hint in the problem helps with that. Then you can start the calculations.

b) If you think you have the basic ideas, but are not sure about certain steps, you should aim to get some partial credit by showing some of the work *clearly*. For example, if you are trying to use some vector $\overrightarrow{PQ}$, then show me how to find it and how you want to use it. Your picture should include P and Q, with their coordinates included. Show your calculation of $\overrightarrow{PQ}$. I cannot give partial credit if your work appears to be random scribbling. Make sure your good thoughts come through.

4) $A = ||\overrightarrow{QP} \times \overrightarrow{QR}||/2 = 7\sqrt{6}/2$.

5) The direction is $\mathbf{v} = \overrightarrow{P_1P_2} = \langle 3, -3, 3\rangle$, and using P_1 as a base point (for $t = 0$), we get

$x = 2 + 3t$
$y = 3 - 3t$
$z = 4 + 3t$

6) $\mathbf{u} = \frac{1}{7}\langle 5, 2, 1\rangle$ and $\mathbf{v} = \frac{1}{7}\langle 8, -1, -4\rangle$. The standard method is the same as for scalars. For example, add 2 times the second equation to the first, etc.

7) Based on geometry, or the hint, a normal vector is $\mathbf{n} = \overrightarrow{PQ} = \langle 0, 2, 6\rangle$. Using R as a base, we get $2(y + 3) + 6(z - 5) = 0$. There are other equivalent answers.

8) TFTTT TFFTF. The worst results were on the 2nd one; consider $\mathbf{w} = -2\mathbf{v}$, for example.

9) These are all in the book and the lectures. Ideally, your proof should include a picture and a couple of sentences. You are trying to explain and convince someone, with more than just a quick calculation (which alone might only be worth about 6-7 points).