MAC 2313 Sept 13, 2018
Exam I Prof. S. Hudson
NAME:

Show all your work and reasoning for maximum credit. Do not use a calculator, book, or personal paper. You may ask about ambiguous questions or for extra paper. Hand in any extra paper you use with your exam. Each problem is 10 points, except #8.

1) Find a vector in R^3 with length $\sqrt{17}$ and with the same direction as $\mathbf{v} = \langle 7, 0, -6 \rangle$.

2) Find the area of the parallelogram with $\mathbf{u} = \langle 2, -2, 4 \rangle$ and $\mathbf{v} = \langle 0, 3, 1 \rangle$ as adjacent sides.

3) Find the equation of the plane through $P(5, 3, 1)$ that is perpendicular to the planes $3x + y + z = 5$ and $x - y + 5z = 2$.

4) The equation $r = 6\cos\theta$ is given in cylindrical coordinates. Express this in rectangular coordinates and sketch the graph. If you cannot get started easily, try multiplying both sides of the given equation by some variable, such as x or r.

5) Determine whether $\mathbf{r}(t) = 3\sin t\mathbf{i} - 2t\mathbf{j}$ is continuous at $t = 0$. Explain your reasoning. For maximum credit, include the Ch.12 definition of limit for vector-valued functions.

6) Find parametric equations for the line tangent to $y = x^2$ at the point $P(2, 4)$.

7) Find an equation for the trace of $x^2 - 4y^2 + z^2 = 13$ in the plane $z = 2$. State what kind of conic section the trace is [is it a parabola, etc ?].

8) [20 pts] Answer True or False. You do not have to explain these. Assume that $\mathbf{u}, \mathbf{v}$ (etc) are arbitrary vectors in R^3 unless stated otherwise.

If $\mathbf{u} \perp \mathbf{v}$ then $(5\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + 2\mathbf{v}) = 5||\mathbf{u}||^2 + 2||\mathbf{v}||^2$.

If $||\mathbf{v}|| < ||\mathbf{v} + \mathbf{w}||$ then $\mathbf{w} \cdot \mathbf{v} > 0$.

If $\mathbf{u} \neq \mathbf{0}$ then there are exactly two unit vectors perpendicular to $\mathbf{u}$.

$\text{proj}_{\mathbf{u}}\mathbf{u} = \mathbf{u}$.

$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$.

$z = 3x^2 + y^2$ is a parabolic hyperboloid.

The parametric equations $x = t^3 + \sin t$ and $y = t^4 \cos t$ can be written in vector form as $\mathbf{r}(t) = (t^4 \cos t)\mathbf{i} + (t^3 + \sin t)\mathbf{j}$.

If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ then $\mathbf{v} = \mathbf{w}$.

If $\text{proj}_{\mathbf{u}}\mathbf{v} = \text{proj}_{\mathbf{v}}\mathbf{u} \neq \mathbf{0}$ then $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$.

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then $|\mathbf{u} \cdot \mathbf{v}| = ||\mathbf{u}||\ ||\mathbf{v}||$.

9) [10 pts] Choose ONE and prove it. You can answer on the back, but if so, leave a note here.

a) State and prove the formula for $\mathbf{u} \cdot \mathbf{v}$ (the one with θ) in R^2.

b) The plane in R^3 through $P(x_0, y_0, z_0)$ with normal vector $\mathbf{N} =< n_1, n_2, n_3 >$, has the equation

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

c) The volume of the parallelpiped that has $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ as adjacent edges is $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

Remarks and Answers: The average, excluding some very low scores, was approx 68 / 100, which is fairly normal. The highest two scores were 90 and 87. The averages for each problem were similar, but low on 4,5,6 (approx 45% to 50%), and high on 2, 9 (approx 85%). The first 5 problems were based on these exercises from the textbook; 11.2.23b, 11.4.17, 11.6.31, 11.8.23 and 12.2.5. Here is an advisory scale for the exam, probably more accurate than the one on the syllabus:

A's 77-100

B's 67-76

C's 57-66

D's 47-56

1) Normalize, then multiply by the desired length, $\mathbf{w} = \sqrt{17/85}\ \langle 7, 0, -6 \rangle$.

2) $A = ||\mathbf{u} \times \mathbf{v}|| = ||\langle -14, -2, 6 \rangle|| = 2\sqrt{59}$. A common mistake was to use the dot product, or a triple scalar product.

3) $\mathbf{n_1} \times \mathbf{n_2} = \langle 6, -14, -4 \rangle$, so the eqn is $6(x - 5) - 14(y - 3) - 4(z - 1) = 0$.

4) Using the hint, get $r^2 = 6r\cos\theta$. From the usual conversion formulas, this becomes $x^2 + y^2 = 6x$. Completing the square, get $(x - 3)^2 + y^2 = 9$. In R^2, this is a circle centered at $(3, 0)$.

Since this is a multi-variable calculus class, and since the problems refers to *cylindrical* coordinates (not polar coordinates), the sketch should be in R^3. It is an infinite cylinder with circular cross sections. Very few people noticed that this is an R^3 problem, so I gave approx 9 points for drawing the correct circle in R^2.

The formula $(x - 3)^2 + y^2 = 9$ is OK without including z. Equations do not have to include all three variables.

5) Yes. Since $\lim_{t \to 0} \mathbf{r}(t) = \cdots = \mathbf{r}(0)$ (calculate both sides) this function satisfies the defn of continuous. To follow the instructions, you should also use the definition of limit, checking that $\lim ||\mathbf{r}(t) - \mathbf{r}(0)|| = 0$.

Since so few people did this, I accepted [for partial credit] explanations based on "both components are continuous". Or, on "$\mathbf{r}'(0)$ exists", though this uses a Ch.12.2 idea on a Ch.12.1 problem, which is generally not OK. If you got 8 or 9 points it was probably for one of these semi-OK answers that omits the definition of limit.

Your answer should include phrases such as "Yes, it is continuous, because . . . " and "by the definition of . . . ", etc. A fairly common mistake was to just write out a few equations with no words, sometimes not even with a yes or no.

6) Let $x(t) = 2+t$ and $y(t) = 4+4t$. Other correct answers are possible. This one has the slight advantage that when $t = 0$ it gives $P(2, 4)$.

Most people got something of the form $\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{v}$ (good!) or the equivalent form above, but most did not choose $\mathbf{v}$ correctly, to point in the direction of the tangent line. You can use $\frac{dy}{dx}(2) = 4$ for this.

For example, use that to get $y = 4+4(x-2)$, a standard formula for the tangent line. Simplify to $y = 4x - 4$. Let $x = t$. Then $y = 4t - 4$. This does not match the previous answer, but it has the same $\mathbf{v}$. Both are correct.

7) Most got $x^2 - 4y^2 = 9$. It is a hyperbola.

Many people said "ellipse" despite the minus sign. Another example: $x^2 + 4y^2 = 9$ has no minus signs (like the equation of a circle) and it is an ellipse. A few people said "hyperbolic paraboloid" (or etc) but that is a quadric surface, not a conic section.

8) TFFTT FFFTT

9a) The proofs were mostly good. The most common problems were gaps in explanation; the meaning of θ (if no picture was included), that $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$, quoting the Law of Cosines, etc.

9b) Almost nobody chose 9b), though it is short.

9c) The proofs were mostly good. A common mistake was to treat h as both a scalar ($V = Ah$) and a vector ($\mathbf{h} = \text{proj}_{v\times x}\mathbf{u}$). This led to more problems. I suggest scalar, with $h = ||\text{proj}_{v\times x}\mathbf{u}||$.