MAC 2313 — Sept 26, 2019

Exam I Key — Prof. S. Hudson

1) [10pts total] Given that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4$, find these

i) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) =$

ii) $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) =$

2) [20 pts] A quadrilateral (a 4-sided polygon lying in a plane in R^3) has corners at $P(1, 5, -2)$, $Q(0, 0, 0)$, $R(3, 5, 1)$ and $S(4, 10, -1)$.

2a) Show that this shape is a parallelogram. Explain your reasoning briefly.

2b) Assuming it is a parallelogram, with Q and S as opposite corners, find its area.

2c) Find an equation for the plane that contains the parallelogram.

3) [10 pts] A projectile is fired from ground level with an initial speed of 20 m/sec, at a 45° angle with the ground. You can treat this as a two-dimensional problem (where x is distance along the ground and y is altitude above the ground). Assume in (3b) that gravity is -9.8 m/s^2, and ignore air resistance, so that $\mathbf{a}(t) = -9.8\mathbf{j} = \langle 0, -9.8 \rangle$.

3a) Find $\mathbf{v}(0)$.

3b) Find a formula for $\mathbf{v}(t)$ by solving an initial-value problem. Use an integral.

4) [10 pts] Find the point on the curve $\mathbf{r}(t) = (5 \sin t)\mathbf{i} + (5 \cos t)\mathbf{j} + 12t\mathbf{k}$ at a distance 13π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

5) [10 pts] Find formulas for $\mathbf{T}$, $\mathbf{N}$ and κ for the space curve $\mathbf{r}(t) = (4 \sin t)\mathbf{i} + (4 \cos t)\mathbf{j} + 3t\mathbf{k}$.

6) [10 pts] Find the vector equation of the tangent line to the curve $\mathbf{r}(t) = 4\cos(t)\mathbf{i} - 3t\mathbf{j}$ at the point $P(2, -\pi)$.

7) [20 pts] Answer True or False. Assume $\mathbf{u}$, $\mathbf{v}$ etc are arbitrary vectors in R^3.

$||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$

If two planes in R^3, with normal vectors $\mathbf{N_1}$ and $\mathbf{N_2}$, do not intersect, then $\mathbf{N_1} \times \mathbf{N_2} = 0$.

If $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$ then $\mathbf{v} \cdot \mathbf{u} = 0$.

If the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \pi/6$ then $||\text{proj}_{\mathbf{v}}\mathbf{u}|| = ||\mathbf{u}|| \cos \theta$.

An ellipsoid can enclose a paraboloid.

The velocity and acceleration vectors of a plane curve $\mathbf{r}(t)$ are always perpendicular.

A circle with radius 5 has smaller curvature than one with radius 2.

$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

Any two planes parallel to the same line L are parallel to each other.

Any two planes perpendicular to the same line L are parallel to each other.

8) [10 pts] Choose ONE and prove it, including enough words. If you answer on the back, leave a note here.

a) Prove that if $\mathbf{r}(t)$ is differentiable and $||\mathbf{r}(t)||$ is constant, then $\mathbf{r}'(t)\perp\mathbf{r}(t)$ for all t.

b) The plane in R^3 through $P(x_0, y_0, z_0)$ with normal vector $\mathbf{N} =< n_1, n_2, n_3 >$, has the equation

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

c) If the angle between two unit vectors is θ, with $0 < \theta < \pi$, then $||\mathbf{u} - \mathbf{v}|| = 2\sin(\theta/2)$. (draw a picture, and explain from there).

Remarks and Answers: The average grade among the top 20 was approx 55 out of 100, which is low. The two highest scores were 84 and 82. The average result on most problems was approx 45% to 65%, except that on (4) it was 40%. Here is an unofficial advisory scale for Exam I, which is lower than the one on the syllabus, and probably more accurate.

A's 65 to 100
B's 55 to 64
C's 45 to 54
D's 35 to 44

The scores on Quiz 1 were similar to the ones on Exam I, so you can use this same scale for that. The scores on Quiz 2 were about 10 points lower. It is hard to set any reasonable scale for that, but as a rough guide, perhaps you could lower this scale about 10 points for your score on it.

1) -4 and 4, based on properties of the T.S.P. If you don't know those well, you might deduce the answers from a well-chosen example. For example, you could set $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$ and $\mathbf{u} = 4\mathbf{k}$, etc.

2a) [6pts] *Show that* generally means to *prove*, not just draw a picture. One simple plan is to show, with a calculation, that two opposite sides are parallel. For example, $\overrightarrow{QP} = \langle 1, 5, -2\rangle$ and $\overrightarrow{RS} = \langle 1, 5, -2\rangle$.

2b) [8pts] Using the norm of a cross product such as $\overrightarrow{QP} \times \overrightarrow{QR}$, from any two adjacent sides, get $A = \sqrt{374}$.

2c) [6pts] Set $\mathbf{n}$ = the cross product from 2b, and get $\mathbf{n} = \langle -15, 7, 10\rangle$ or some scalar multiple of that. So, with base point Q, for example, get this (simplification is optional):

$$-15(x - 0) + 7(y - 0) + 10(z - 0) = 0$$

3a) $\mathbf{v}(0) = \langle 10\sqrt{2}, 10\sqrt{2}\rangle$, based on a picture and simple trig ($\cos 45° = \sqrt{2}/2$ etc). A common mistake in 3a and 3b was to answer with a scalar.

3b) $\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \mathbf{a} = \cdots = \langle 10\sqrt{2}, 10\sqrt{2} - 9.8t\rangle$

4) The plan is: location = $\mathbf{r}(t)$ when $s(t) = 13\pi$. To solve this for t, we know $s = \int_0^t ||\mathbf{r}'||$. Compute $\mathbf{r}'$ and get $||\mathbf{r}'|| = 13$. So $s(t) = 13t$ and $t = \pi$. Answer = $\mathbf{r}(\pi) = \langle 0, -5, 12\pi \rangle$.

This is a slight variation on 13.3.9.

5) $\mathbf{r}'(t) = \langle 4\cos t, -4\sin t, 3\rangle$ and $||\mathbf{r}'(t)|| = 5$. So $\mathbf{T} = \frac{1}{5}\langle 4\cos t, -4\sin t, 3\rangle$. $\mathbf{T}' = \frac{1}{5}\langle -4\sin t, -4\cos t, 0\rangle$. Normalizing, $\mathbf{N} = \langle -\sin t, -\cos t, 0\rangle$ and $\kappa = \frac{4}{25}$.

This is a slight variation on 13.5.9.

6) Solve for t by setting $\mathbf{r}(t) = \langle 2, -\pi\rangle$ and get $t = \pi/3$. Then $\mathbf{v} = \langle -4\sin(\pi/3), -3\rangle = \langle -2\sqrt{3}, -3\rangle$. Answer: $\mathbf{q}(t) = \langle 2, -\pi\rangle + t\langle -2\sqrt{3}, -3\rangle$ based on the usual formula for a line, $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$. I changed $\mathbf{r}$ to $\mathbf{q}$ since $\mathbf{r}$ already has another meaning in this example.

7) TTFTF FTFFT

8) Options a and b are in the text and were in the lectures. The proof of b is at the top of page 746. The central ideas are that $\mathbf{N} \perp \overrightarrow{PQ}$ so that $\mathbf{N} \cdot \overrightarrow{PQ} = 0$, and that $\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0\rangle$. These formulas should be explained as part of your answer. The words and notation may vary, but these two ideas are needed.

The sketch for Part c is an isosceles triangle bisected into two right triangles by a line segment between the given two vectors. The rest is basic trig.