MAC 2313 Feb 6, 2019
Exam I Prof. S. Hudson

1) [10 pts] Find the distance from the point $P(1,1) \in R^2$ to the line $y = 2x + 1$ using a projection.

2) [10 pts] Find the area of the triangle with vertices $P(1,3,5)$, $Q(0,0,0)$ and $R(3,2,1)$.

3) [10 pts] Show that the lines L_1 and L_2 intersect and find the point of intersection.
L_1 is $x = 2 + t$, $y = 1 + 2t$ and $z = 3t$.
L_2 is $x = 1 + t$, $y = 3t - 4$ and $z = 2t$.

4) [6 pts] Find the equation of the sphere centered at $P(2,-1,1)$ that is tangent to the plane $y = 2$.

5) [6 pts] Find the equation of the plane through $(3,2,5)$ that is parallel to the plane $4x + 2y - z = 13$.

6) [8 pts] Find an equation of the plane $z = 3$ in spherical coordinates.

7) [10 pts] Find the vector equation of the line tangent to the graph of $\mathbf{r}(t) = \langle t^2, \frac{-1}{1+t}, 4-t^2\rangle$ at the point $P_0(4,1,0)$ on the curve.

8) [10 pts] Compute the unit inward normal vector $\mathbf{N}(t)$ for the helix with $x = 3\cos(t)$, $y = 3\sin(t)$, $z = 4t$.

9) [20 pts] Answer True or False. Assume $\mathbf{v}$ (etc) are arbitrary vectors in R^3 unless stated otherwise.

If two planes in R^3 with normal vectors $\mathbf{N_1}$ and $\mathbf{N_2}$ do not intersect then $\mathbf{N_1} \cdot \mathbf{N_2} = 0$.

$||\mathbf{u} + 2\mathbf{v}|| \leq ||\mathbf{u}|| + 2||\mathbf{v}||$

If $\mathbf{u} \perp \mathbf{v}$, then $||\mathbf{v}|| \leq ||\mathbf{u} + \mathbf{v}||$.

$\mathrm{proj}_{4\mathbf{u}}\mathbf{w} = \mathrm{proj}_{3\mathbf{u}}\mathbf{w}$

$\mathrm{proj}_{\mathbf{i}}\mathbf{w} + \mathrm{proj}_{\mathbf{j}}\mathbf{w} + \mathrm{proj}_{\mathbf{k}}\mathbf{w} = \mathbf{w}$.

An elliptical paraboloid is a surface of revolution.

There are more than two unit vectors normal to any given plane in R^3.

Five different unit vectors cannot be orthogonal to the vector $\mathbf{k}$ in R^3.

$\int_0^4 t^3\mathbf{u}\, dt = (\int_0^4 t^3\, dt)\mathbf{u}$ [Typo corrected 2/13/19].

$\lim_{t\to 0}\langle \frac{t}{2}, \frac{t}{t}, \frac{2t}{t}\rangle = \langle 0,1,2\rangle$

10) [10 pts] Choose ONE and prove it. You can answer on the back, but if so, leave a note here.

a) State and prove the formula for $\mathbf{u} \cdot \mathbf{v}$ (the one with θ) in R^2.

b) The plane in R^3 through $P(x_0, y_0, z_0)$ with normal vector $\mathbf{N} =< n_1, n_2, n_3 >$, has the equation

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

c) The volume of the parallelpiped that has $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ as adjacent edges is $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

Bonus) [about 5 pts, maybe not easy] A projectile is fired from height 0, at an initial speed of 20 m/sec, at a 45° angle with the ground. Assume gravity is -9.8 m/s^2. How long until the object hits the ground ? Solve this using Calculus (perhaps with initial value problems) not with memorized physics formulas.

Remarks and Answers: The average among the top 30 was approx 63%, which is normal but a little low. The top two scores were 101 and 100. The average on problems 2 and 5 was over 80%, but was under 50% on problem 8. Here is an advisory scale for Exam I:

A's 71 to 100
B's 61 to 70
C's 51 to 60
D's 41 to 50

1) $\frac{2}{\sqrt{5}}$. Find a point on L such as $A(0, 1)$ and a normal vector for L such as $\mathbf{n} = \langle 2, -1 \rangle$ (very similar to finding one for a plane in R^3). Project $\overrightarrow{AP} = \langle 1, 0 \rangle$ onto $\mathbf{n}$. The distance is the absolute value of the scalar projection (and the norm of the vector projection), $\frac{2}{\sqrt{5}}$.

Alt.solution (this is a bit longer but is recommended for similar problems in R^3 where $\mathbf{n}$ makes less sense). Find a second point on L such as $B(1, 3)$ so the direction of L is $\mathbf{v} = \langle 1, 2 \rangle$. Project $\overrightarrow{AP}$ onto $\mathbf{v}$ to get a vector $\mathbf{p} = (1/5)\langle 1, 2 \rangle$. The distance is the norm of $\overrightarrow{AP} - \mathbf{p}$, the same answer.

For either solution, a picture (including the projection vector) is highly recommended to guide your calculations. I did not give much credit if you did not use a projection.

2) $A = \frac{1}{2}||\overrightarrow{QP} \times \overrightarrow{QR}|| = \frac{7\sqrt{6}}{2}$. Two students noticed that there is a right angle at R, so that you can use $A = bh/2$ as a shortcut. Of course, this is not a reliable general-purpose method.

3) They intersect when $t_1 = 2$ and $t_2 = 3$, at the point $P(4, 5, 6)$.

4) $(x - 2)^2 + (y + 1)^2 + (z - 1)^2 = 9$. The distance from the center to the plane is 3 (from y= -1 to y=2).

5) $4(x - 3) + 2(y - 2) - (z - 5) = 0$.

6) $\rho \cos \phi = 3$ (because $z = \rho \cos \phi$). It is OK to add that $0 \leq \phi < \pi/2$, but this is not necessary.

Reminder - I cannot give full credit for a random mix of equations with no clear ending, even if one of them is OK. Clearly identify your answer by circling it, or underlining it, or by explaining your logic.

7) Since the letter $\mathbf{r}$ is taken, we can write the answer as $\mathbf{s}(t) = \langle 4, 1, 0\rangle + t\langle -4, 1, 4\rangle$. For the velocity vector in this answer, notice that $\mathbf{r}(-2) = \langle 4, 1, 0\rangle$ and $\mathbf{r}'(-2) = \langle -4, 1, 4\rangle$.

8) $\mathbf{N}(\mathbf{t}) = \langle -\cos(t), -\sin(t), 0\rangle$. This is a slightly long but straightforward calculation based on the definitions. See 12.4, Ex.2.

9) FTTTT FFFTT (see me if needed)

10) See the text or lecture notes. Most people chose (a) and did OK. The main problems were explanation gaps. Mention the law of cosines, explain what θ is (or draw a picture), etc.

B) $t = \frac{20\sqrt{2}}{9.8}$ secs