MAC 2313 | Sept 24, 2020
Exam I Key | Prof. S. Hudson

1) (10 pts) Give a geometric description of the set of points in space whose coordinates satisfy the pair of equations: $z = 0$ and $y = 0$. For example (but not correct) "It is a sphere of radius 3 in the first octant, centered at (4,5,6)".

2) (10 pts) Find a formula for $\mathbf{r}(t)$ given $\mathbf{r}'(t) = (180t)\mathbf{i} + (180t - 16t^2)\mathbf{j}$ and $\mathbf{r}(0) = 100\mathbf{j}$.

3) (5 pts + EC) Suppose $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}$ and $||v(t)|| = t$ and $||a(t)||^2 = t^2 + 1$. Find $a_T(t)$ and $a_N(t)$ and simplify both. For a little extra credit, find all 5 parts of the Ch.13.5 formula $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ and confirm it.

$a_T =$

$a_N =$

4) [10 pts] Find two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^3 such that
$\mathbf{u} + \mathbf{v} = 3\mathbf{i} + \mathbf{k}$ and
$3\mathbf{u} - \mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

5) [12 pts] Find the shortest distance from the point $P(1, 2, 3)$ to the plane T where $4(x - 2) - 3(y - 1) + 5z = 0$. Perform these steps to do that. A rough sketch might help.

a) Find a normal vector $\mathbf{n}$ to T.

b) Find some point Q in T and compute the vector $\mathbf{v} = \overrightarrow{QP}$ from Q to P.

c) Compute the projection vector $\mathbf{p} = \text{proj}_{\mathbf{n}}\mathbf{v}$ [or compute the scalar projection].

d) Use this work to find the distance from P to T.

6) [10 pts] Find the arc length of the space curve $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), \sqrt{5}t\rangle$, with $0 \le t \le \pi$.

7) [21 pts] Answer True or False. Assume $\mathbf{v}$ (etc) are arbitrary vectors in R^3 unless stated otherwise.

$\int \cos^2 x \, dx = x + \frac{\sin(2x)}{2} + C$
The graph of $x + y^3 + z^2 = 3$ is a quadric surface.
The graph of $z = x^2 - y^2$ is a quadric surface with a saddle point.
If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors in R^3 then $||\mathbf{u} \times \mathbf{v}|| \le 1$.
If $\mathbf{u} = \overrightarrow{OP}$ points into the second quadrant of R^2 then $\mathbf{u} \cdot \mathbf{i} \le 0$.
If $\mathbf{r}(t)$ is differentiable then it is a smooth curve.
The formula for radius of curvature is $\rho = \frac{1}{\kappa}$.

8) [12 pts] Let $\mathbf{r}(t) = \langle e^t \cos(t), e^t \sin(t), 2\rangle$. Find $\mathbf{T}(t)$, $\mathbf{N}(t)$ and κ for this space curve.

9) [10 pts] Choose ONE and prove it. This means formulas and *words*, probably no examples.

a) State and prove the formula for $\mathbf{u} \cdot \mathbf{v}$ (the one with θ) in R^2.

b) The volume of the parallelpiped that has $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ as adjacent edges is $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$.

Remarks, Scale, Answers: The average among the top 20 scores was approx 64%. The high scores were 92 and 90. The problems with the best average were #2 (92%) and #6 (86%), the lowest average was on #3 (30%). A rough scale for the exam is

A's 72 to 100
B's 62 to 71
C's 52 to 61
D's 42 to 51

A rough scale for the semester so far, not yet including HW, is below. Average your three quiz scores at 2 points each with your exam score at 15 points and use the scale. For example suppose your quiz scores were 40, 50 and 60, and that your exam score was 70. Your average is $(40 \cdot 2 + 50 \cdot 2 + 60 \cdot 2 + 70 \cdot 15)/21 = 64.3$, which is a B, almost a B+.

A's 69 to 100
B's 59 to 68
C's 49 to 58
D's 39 to 48

1) It is the x-axis. Other good descriptions were possible, but not very common. For full credit they should be clear and error-free. The results on (1) and (2) were mostly good.

2) $\mathbf{r}(t) = 90t^2\mathbf{i} + (90t^2 - 16t^3/3 + 100)\mathbf{j}$. Like most IVP's we integrate $\mathbf{r}(t)$, include the unknown $\mathbf{C}$, and deduce that $\mathbf{C} = 100\mathbf{j}$.

This is an exercise in the textbook. It is a little ambiguous whether the setting is R^2 or R^3. I assumed R^2 but the answer above is correct either way.

3) The results were mostly weak, probably because 13.5 was not yet in the graded homework. But this section was on the exam list and in recent lectures. The problem comes from 13.5 Ex.1, with some of the work given. The main ideas are

$a_T = \frac{d}{dt}||\mathbf{v}|| = 1$ and
$||\mathbf{a}||^2 = a_T^2 + a_N^2$ so $a_N = t$.

4) This is just like solving a 2x2 system for x and y. Add the two equations, and then divide by 4 to get $\mathbf{u} = \langle 1, 1/4, 0\rangle$. Plug this in and solve for the other vector, $\mathbf{v} = \langle 2, -1/4, 1\rangle$. And as usual show the work.

5a) $\mathbf{n} = \langle 4, -3, 5\rangle$.

b) Assuming you found $Q = (2, 1, 0)$ (the obvious point), then $\mathbf{v} = \overrightarrow{QP} = \langle -1, 1, 3\rangle$.

c) $\text{proj}_{\mathbf{n}}\mathbf{v} = \frac{\mathbf{n}\cdot\mathbf{v}}{||\mathbf{n}||^2}\mathbf{n} = \frac{8}{50}\langle 4, -3, 5\rangle$. The norm of this is the scalar projection, $\alpha = \frac{8}{\sqrt{50}}$.

d) The distance from P to T is $\alpha = \frac{8}{\sqrt{50}}$. The reason is pretty clear from a picture, even an inaccurate ("formal") one.

6) Compute $||\mathbf{r}'|| = 3$ so $L = \int_0^\pi 3\ dt = 3\pi$.

7) FFTTTFT For explanations, just ask me or our LA.

8) Many answers were on the right track, but many had calculation errors. Many were too disorganized to grade accurately (suggestions: circle your answers; use the back if you need room). Some answers had the wrong form (vector in R^3, vector in R^3, scalar).

8a) $\mathbf{T}(t) = \frac{1}{\sqrt{2}}\langle \cos t - \sin t, \cos t + \sin t, 0\rangle$. For this, compute $\mathbf{r}'(t)$ using the product rule, then normalize it. Simplify, so that the e^t drops out and 8b is easier. Suggestion: give extra care to the first steps of any 3-part problem.

8b) $\mathbf{N}(t) = \frac{1}{\sqrt{2}}\langle -\cos t - \sin t, \cos t - \sin t, 0\rangle$. Suggestion: check that your answers to 8a and 8b are unit vectors and are orthogonal to each other. Do it in your head if you can.

8c) $\kappa(t) = \frac{1}{\sqrt{2}e^t}$. Since $||\mathbf{T}'(t)|| = 1$, $\kappa(t) = \frac{1}{||\mathbf{r}'||}$ which you already calculated in 8a.

9) See the text for two good proofs. Most people chose 9a and most answers were pretty good. For full credit you needed good organization and explanation, including most of these items (for 9a):

State the formula you are going to prove at the start. Explain that the vectors are in R^2 and what θ is. Include a picture, with the triangle you will use in the proof. Quote the Law of Cosines to explain the first formula in your proof. Mention that $||\mathbf{u}||^2 = \mathbf{u}\cdot\mathbf{u}$ and $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ to justify your simplifications. A good proof might omit some of these, but in general the more you explain, the better.