MAC 2313 Feb 10, 2021
Exam I Prof. S. Hudson

---

1) Find the component form of the vector obtained by rotating the unit vector $\langle 0, 1\rangle$ 120 degrees counterclockwise about the origin. A picture is strongly recommended.

2) Let $\mathbf{u} = \langle 0, 3, 4\rangle$ and $\mathbf{v} = \langle 6, 0, 8\rangle$. Compute $\text{proj}_{\mathbf{v}}\mathbf{u}$ and simplify.

3) Find an equation for the plane that is perpendicular to the plane $3x + y - 2z = 1$, and contains the line $L : x = 2 + t$, $y = 1 + 5t$, $z = 1 + 4t$.

4) Sketch the hyperboloid $z^2 - x^2 - y^2 = 1$ in 3D including the trace where $x = 0$ and at least one other trace.

5) Find parametric equations for the line tangent to the circle $x^2 + y^2 = 25$ at $P(3, -4)$. You can compute $\frac{dy}{dx}$ with implicit differentiation, or observe that the tangent line must be perpendicular to the radius at P.

6) Find the unit tangent vector $\mathbf{T}(t)$ and the arc length L of the curve $\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k}$, $0 \le t \le 8$.

7) Find the curvature $\kappa(t)$, given $\mathbf{r}(t) = t\ \mathbf{i} + 3\sin t\ \mathbf{j} + 3\cos t\ \mathbf{k}$. Various methods are OK, but show your work clearly.

8) [20 pts] Answer True or False. The vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ have the usual meaning, but the others, $\mathbf{u}$, $\mathbf{v}$ (etc) are arbitrary vectors in R^3.

$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$ describes a circular helix.

If two intersecting planes in R^3 have normal vectors $\mathbf{n}$ and $\mathbf{m}$ then the angle between the planes is the angle between $\mathbf{n}$ and $\mathbf{m}$.

The surfaces $x + y + z = 0$ and $x^2 + y = 4$ intersect in a curve with parametric equations, $\langle t, 4 - t^2, t^2 - t - 4\rangle$

If $\mathbf{u} \perp \mathbf{v}$, then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

If $P(x_0, y_0, z_0)$ lies in the xy-plane and in the yz-plane, then $y_0 = 0$.

$\mathbf{k} \times \mathbf{j} = \mathbf{i}$

$||\mathbf{v} + \mathbf{v}|| = 2||\mathbf{v}||$

$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{u}) = \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$.

In section 13.5 (Motion) the formula for a_T is $\frac{d^2s}{dt^2}$.

The graph of $x^2 + y^2 - z^2 = 1$ is a quadric surface.

9) [10 pts] Choose ONE and prove it. This means formulas and *words*, probably no examples.

a) State and prove the formula for $\mathbf{u} \cdot \mathbf{v}$ (the one with θ) in R^2.

b) The plane in R^3 through $P(x_0, y_0, z_0)$ with normal vector $\mathbf{N} =< n_1, n_2, n_3 >$, has the equation

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Bonus) Find a vector $\mathbf{u} \in R^3$ such that these are all true:

$\text{proj}_{\mathbf{i}}\mathbf{u} = 3\mathbf{i}$, $\text{proj}_{\mathbf{j}}\mathbf{u} = \mathbf{0}$ and $||\mathbf{u}|| = 5$.

Remarks, Scale, Answers: The average among the top 80% or so was 67 / 100, which is fairly normal. The two best scores were 92 and 91. The best average results were on problems 2 and 4 (83% ish). The lowest average results were on problem 3 and 5 (43% ish). Approx 7 people got some credit on the Bonus, which is a pretty good number.

Here is an advisory scale for the Exam. The scale for the semester so far, including the quizzes and HW, is probably about the same, since the exam counts more than the rest.

A's 75 - 100
B's 65 - 74
C's 55 - 64
D's 45 - 54

1) $\langle -\sqrt{3}/2, -1/2 \rangle$

2) $\frac{32}{100}\langle 6, 0, 8 \rangle = \langle \frac{48}{25}, 0, \frac{64}{25} \rangle$.

3) One point on the plane is $\mathbf{r}(0) = \langle 2, 1, 1 \rangle$. Next, we need a normal vector $\mathbf{n}$ with $\mathbf{n} \perp \langle 3, 1, -2 \rangle$ and $\mathbf{n} \perp \langle 1, 5, 4 \rangle$ (check my reasoning!). Use the cross-product and get $\mathbf{n} = \langle 14, -14, 14 \rangle$ or use simply $\mathbf{n} = \langle 1, -1, 1 \rangle$. One equation is $(x - 2) - (y - 1) + (z - 1) = 0$.

4) The trace where $x = 0$ is a hyperbola in the yz-plane. The trace where $z = 2$ (or 3, etc) is a horizontal circle above the xy-plane. There are no points on the xy-plane, where $z = 0$. Putting the info together you should get a hyperboloid of two sheets. I won't try to insert a picture on this key. There are pictures in the book and it is not too hard to create them from free software.

5) $\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{v} = \langle 3, -4 \rangle + t\langle 4, 3 \rangle$, so the PE's are $x = 3 + 4t$ and $y = -4 + 3t$. As usual, other answers are possible. I got the direction $\langle 4, 3 \rangle$ based on the usual 'trick', $\langle 3, -4 \rangle \cdot \langle 4, 3 \rangle = 0$. You could also get it by calculating that $\frac{dy}{dx} = 3/4$ but that takes longer.

6) $\mathbf{r}'(t) = \langle 1, 0, t^{1/2}\rangle$ so $||\mathbf{r}'(t)|| = (1+t)^{1/2}$ so

$||\mathbf{T}(t)|| = (1+t)^{-1/2}\langle 1, 0, t^{1/2}\rangle$. Then

$L = \int_0^8 (1+t)^{1/2}\ dt = (2/3)(1+t)^{3/2}|_0^8 = (2/3)(27-1) = 52/3$.

7) This problem is from the worksheet on Monday, Feb 8. The 13.4 method on the answer key is good, though there is a small error in the last step. Ch. 13.4 contains a very similar Example with the same method. Here is another good method from Ch.13.5:

$\mathbf{v} = \mathbf{r}'(t) = \langle 1, 3\cos(t), -3\sin(t)\rangle$
$\mathbf{a} = \langle 0, -3\sin(t), -3\cos(t)\rangle$
$\mathbf{v} \times \mathbf{a} = \langle -9, 3\cos(t), -3\sin(t)\rangle$.

Since I am not perfect, I checked that $\mathbf{v} \times \mathbf{a} \perp \mathbf{v}$ and that $\mathbf{v} \times \mathbf{a} \perp \mathbf{a}$ (and you might consider more checking). Answer:

$\kappa = \frac{||\mathbf{v}\times\mathbf{a}||}{||\mathbf{v}||} = \cdots = \frac{3}{10}$.

8) TTTFF FTTTT. On the 8th one, notice that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{u}) = \mathbf{0}$ and $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$.

9) See the text.

B) $\mathbf{u} = \langle 3, 0, 4\rangle$. If you check for yourself that this answer works, you will probably also see the methods I used. Another correct answer is $\mathbf{u} = \langle 3, 0, -4\rangle$.