1) Short Answer: 5 pts each:

a) \( \lim_{(x,y,z) \to (2,8,1)} \sqrt{xy} \tan\left(\frac{3\pi z}{4}\right) = \)

b) Find \( f(x, y) \) such that \( f_x(x, y) = 2xy^3 + 1 \) and \( f_y(x, y) = 3x^2y^2 \).

c) Find every point on the surface \( z = f(x, y) \) where the tangent plane is horizontal; \( z = x^2 + 4x + y^3 \).

d) Find \( \partial w/\partial s \), given that \( w = pq \sin r, p = 2s + t, q = s - t \) and \( r = st \).

e) Find the directional derivative of \( f(x, y) = e^x \sin y \) at \( P(0, \pi/4) \) in the direction of \( \mathbf{v} = \langle 1, -1 \rangle \).

2) [10pts] Use the exact value of \( f(P) \) and \( df \) to approximate \( f(Q) \), given that \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \), \( P(3, 4, 12) \), \( Q(3.03, 3.96, 12.05) \).

3) [10pts] Let \( f(x, y) = 2x^2 + 3xy + 4y^2 \). Find the maximal directional derivative at \( P(1, 1) \) and the direction in which it occurs.

4) [10pts] Find the first octant point \( P(x, y, z) \) on the surface \( x^2y^2z = 4 \) closest to the origin. Suggestion: minimize the square of the distance.

5) [10pts] Find the derivative matrix for the polar coordinate transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \), given by \( x = r \cos \theta \) and \( y = r \sin \theta \).

6) [10pts] Compute the double integral \( \int_1^2 \int_0^\pi 2xy + \sin(x) \, dx \, dy \)

7) [15pts] Answer True or False. You do not have to explain.

The Two-Variable Second Derivative Test is inconclusive if \( \Delta = 0 \).

\[
\int_1^2 \int_3^4 yx^2 \, dx \, dy = \int_3^4 \int_1^2 yx^2 \, dy \, dx
\]

If \( \nabla f(a, b) = 0 \) and \( f_{xx} = f_{yy} < 0 \) at \( (a,b) \), then \( f \) has a local maximum there.
If $\nabla f(a,b)$ exists, then $f$ is differentiable at $(a,b)$.

Most of the level curves of $f(x,y) = \exp(-x^2 - y^2)$ are circles.

8) [10pts] Prove ONE:

A) The Lagrange Multiplier Theorem ($\nabla f(P) = \lambda \nabla g(P)$).

B) If $n$ is a positive integer and all the derivatives exist, then $\nabla (u^n) = nu^{n-1}\nabla u$.

Answers and remarks: As of 11/5/05 the average is around 65/100. So, each letter grade interval is about 3 points lower than planned; eg the A/B borderline is around 77. The average grade was good on problems 1, 2 and 6, but bad on 4 (not hard!), 5 and 8. Exam 3 may contain another problem like 4.

1a) [see 12.3-15] -4

1b) [revised 12.4-41] $f(x, y) = x^2y^3 + x$ (you can also include ”+C”)

1c) [12.5-7] Set $f_x = f_y = 0$ to find $x$ and $y$, but don’t forget that a point on the surface has a $z$ coordinate, too. So, (-2,0,-4).

1d) [12.7-6] I accepted $2q \sin r + p \sin r + tpq \cos r$ (from the Chain Rule), but you can plug-in for $p$ (etc), if you want. Another acceptable method (but not recommended, because it avoids the lesson) is to plug-in for $p$ (etc) at the start.

1e) [12.8-12] 0

2) [12.6-21] $f(P) = \sqrt{9 + 16} = 14 = 13$ and $df = f_x \Delta x + f_y \Delta y + f_z \Delta z = 3(.03)/13 + 4(-.04)/13 + 12(.05)/13 = .53/13$ (this simplification is optional). So, $f(Q) \approx 13 + .53/13$.

3) [12.8-21] $\nabla f(P) = \langle 7, 11 \rangle$, so the max d.d. is $||\nabla f(P)|| = \sqrt{170}$ and the dir is $170^{-1/2}\langle 7, 11 \rangle$. 

2
Some students got the two questions reversed (e.g., they said the max d.d. was $170^{-1/2}(7,11)$) and only got partial credit. Likewise, those who did not label their answers at all lost some points. It is usually wise to include a few words in each of your answers. Eventually, I decided to accept $(7,11)$ as the direction, though I prefer the unit vector.

4) [Exercise 33 from 12.5] Set $f(x,y,z) = x^2 + y^2 + z^2$ and use the constraint to get $x^2 = 4/(y^2z)$ and $f(x,y,z) = 4/(y^2z) + y^2 + z^2$. Then set $f_y = f_z = 0$ and after some algebra, get $P(\sqrt{2}, \sqrt{2}, 1)$. You can solve for $z$ instead of $x$, of course (but it’s a little messier that way).

Many many people did not get off to a good start on this one, so I did not try to give much partial credit - unless the basic method was quite clear and correct. You were not supposed to use Lagrange multipliers (Ch.12.9) but I gave full credit if you got the correct answer that way.

5) See example 10 of Ch 12.7, which we did in class.

6) $\pi^2/2 + 2$

7) TTFFT (ask)

8a) See textbook. For full credit, your proof should include all the key ideas, such as using the implicit function theorem and $\nabla g(P) \neq 0$ to get $r(t)$, using $F'(t_0) = 0$ to get info about $f$, etc.

8b) By the Calc I chain rule, $D_1u^n = nu^{n-1}D_1u$ and the same goes for $D_2$. Then

$$\nabla u^n = \langle D_1u^n, D_2u^n \rangle = \langle nu^{n-1}D_1u, nu^{n-1}D_2u \rangle = nu^{n-1}\nabla u$$

Part b was intended to be easier, because it was not advertised as well.