MAC 2313 Oct 16, 2017
Exam II Prof. S. Hudson

1) [10 pts] Choose ONE:

a) State the definition of $f(x,y)$ is *differentiable* at $P(x_0, y_0)$.

b) Show that the graph of $\mathbf{r}(t) = \sin(t)\mathbf{i} + 2\cos(t)\mathbf{j} + \sqrt{3}\sin(t)\mathbf{k}$ is a circle. Find its center and radius. Hint: Show that the curve lies on both a sphere and a plane.

2) [10 pts] Find the arc length parametrization of the circular helix $\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$ that has reference point $\mathbf{r}(0) = (1, 0, 0)$ and has the same orientation as the given helix.

3) [10 pts] Determine whether the limit exists. If so, find its value.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{3x^2 + 2y^2}$$

4) [10 pts] Let $f(x,y) = 2xy^2$, $P(1,2)$ and $Q(0.99, 2.02)$. Use a total differential to approximate the change in the value of f from P to Q. Simplify completely.

5) [10 pts] Find the curvature $\kappa(t)$ of the curve $x = 1 - t^3$ $y = t - t^2$. For full credit, use the formula for $\kappa(t)$ with the cross product (setting $z = 0$ if needed).

6) [10 pts] A point moves along the intersection of the plane $y = 3$ with the surface $z = \sqrt{29 - x^2 - y^2}$. At what rate is z changing with respect to x when the point is at $(4, 3, 2)$?

7) [20 pts] Answer T or F. You do not have to justify your answers:

If f_x and f_y exist at P, then $f(x,y)$ is differentiable at P.

If f_x and f_y exist at P, then $f(x,y)$ is continuous at P.

If the line $x = 3$ is a contour of $f(x,y)$ through $(3,5)$ then $f_y(3,5) = 0$.

If f_{xy} and f_{yx} exist at P, then they are equal at P.

By definition, the binormal vector is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

For a moving particle, the unit tangent vector and the velocity vector are parallel.

If C is any smooth curve in R^2, then both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ exist for all t in the domain.

If $||\mathbf{r}(t)|| = 3$ for all t, then $\mathbf{r}'(t) \perp \mathbf{r}''(t)$ for all t.

If $\frac{ds}{dt} = 3$ for all t, then $\mathbf{r}'(t) \perp \mathbf{r}''(t)$ for all t.

For a particle moving from time a to time b, $||\mathbf{r}(b) - \mathbf{r}(a)|| \leq s =$ distance traveled.

8) [10 pts] Let $\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$. Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and the equation of the normal plane when $t = \pi/4$.

9) [10 pts] Choose ONE:

A) State and prove Theorem 12.6.3, the first part only, the formula for a_T that includes $\mathbf{v}$ and $\mathbf{a}$. Include a relevant picture, enough formulas and enough words.

B) An automobile travels at a constant speed around a curve whose radius of curvature is 1000m. What is the maximum allowable speed if the maximum acceptable value for the normal scalar component of acceleration is 1.6m $/s^2$?

Remarks and Answers: The average was 68 / 100, with high scores of 94 and 87. The results were a bit low on problems 1, 3 and 7 (approx 55%) and a bit high problems 4 and 6 (approx 80%). Here is an advisory scale for the exam:

A's 76 to 100
B's 66 to 75
C's 56 to 65
D's 46 to 55

Here is an advisory scale for the semester so far (average your two exam scores and use that):

A's 74 to 100
B's 64 to 73
C's 54 to 63
D's 44 to 53

1) For 1a) see the text or the lectures (where I abbreviated the numerator as $\Delta f - df$).

For 1b), check that $||\mathbf{r}(\mathbf{t})|| = 2$ for all t, which shows the curve lies on a sphere of radius 2 centered at the origin. Also, since $z = \sqrt{3}x$, it lies in a plane through the origin. So, it is a circle (I did not expect more justification than this). The center and radius are the same as for the sphere.

2) Calculate that $||\mathbf{r}'(t)|| = \sqrt{5}$ for all t, so $s = \sqrt{5}t$ and $\mathbf{r} = \cos(\frac{s}{\sqrt{5}})\mathbf{i} + \sin(\frac{s}{\sqrt{5}})\mathbf{j} + \frac{2s}{\sqrt{5}}\mathbf{k}$.

3) L=0. The best way is polar coordinates, $\lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^2(3\cos^2\theta+2\sin^2\theta)} = 0$. The important part is $\lim_{r\to 0}\frac{r^3}{r^2} = 0$.

4) The question is to compute $df = f_x dx + f_y dy = 2y^2(-0.01) + 4xy(0.02) = 0.08 - 0.16 = -0.08$.

Note that $dx = \Delta = x - x_0 = 0.99 - 1.00$ and not the other way around. In general, x_0 will be the simpler number (1.00), and the phrase 'from P to Q' also indicates how to do the subtraction.

5) $\kappa = \frac{||\mathbf{r}'\times\mathbf{r}''||}{||\mathbf{r}'||^3} = \frac{|6t-6t^2|}{(9t^4+4t^2-4t+1)^{3/2}}$.

Most people remembered the formula correctly, but then there were very many calculation errors. Get $\mathbf{r}' \times \mathbf{r}'' = (6t - 6t^2)\mathbf{k}$ (most people got this, but not the easier next step), so $||\mathbf{r}' \times \mathbf{r}''|| = |6t - 6t^2|\ ||\mathbf{k}|| = |6t - 6t^2|$. Likewise, $||\mathbf{r}'|| = \sqrt{(-3t^2)^2 + (1-2t)^2}$ etc should be easy.

6) -2.

7) FFTFT TFFTT. The 7th one is maybe a little tricky, but $\mathbf{N}$ can fail to exist. Consider a straight line, for example.

8) Get $\mathbf{T} = \langle -\sin(t), \cos(t), 0\rangle = \langle -\sqrt{1/2}, \sqrt{1/2}, 0\rangle$ and $\mathbf{N} = \langle -\cos(t), -\sin(t), 0\rangle = \langle -\sqrt{1/2}, +\sqrt{1/2}, 0\rangle$ fairly easily, since $t = s =$ arc length. For the normal plane (which contains the *normal* and bi*normal* vectors), let $\mathbf{n} = \mathbf{T}$ and get

$$-\sqrt{1/2}(x - \sqrt{1/2}) + \sqrt{1/2}(y - \sqrt{1/2}) = 0.$$

This simplifies to $x = y$, but I did not require that last step.

9a) See the text or lectures. The most common problem was with the step $a_T = ||\mathbf{a}|| \cos\theta$. This is correct, but it makes no sense without a picture, and/or a good explanation.

9b) Recall that $a_N = (ds/dt)^2\kappa$. So, $(ds/dt)^2\kappa \leq 1.6$ and $\kappa = 1/1000$, so speed $= ds/dt \leq \sqrt{1600} = 40$ m/s.