MAC 2313 Oct 11, 2018
Exam II Key Prof. S. Hudson

1) [15pts] Let $\mathbf{r}(t) = \langle 3\sin(t), 3\cos(t), 4t\rangle$. Compute $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$. Then, find the equation of the osculating plane.

2) [5 pts each] 2a) Suppose $T(x, y) = xy$ is the temperature at (x, y) on a thin plate which occupies the first quadrant of the plane. Sketch the level curves (the isothermal curves) where $T = 2$ and $T = 3$.

2b) Give an example of a function $f(x, y)$ such that $f_x(0, 0) = f_y(0, 0) = 0$, which is not continuous at $(0, 0)$.

2c) Suppose $z = f(x, y)$ has a local linear approximation (a tangent plane) at the point $(3, 2)$. Suppose the approximation has the equation $z = 4(x - 3) + 5(y - 2) + 6$. Compute $\nabla f(3, 2)$.

3) Let $\mathbf{r}(t) = \langle t^3 - 2t, t^2 - 4\rangle$. Compute the scalar components of acceleration, a_T and a_N, when $t = 1$.

4) Solve the initial value problem: $\mathbf{y}'(t) = 3t\mathbf{i} - 2t^2\mathbf{j}$ with $\mathbf{y}(0) = \mathbf{i} + \mathbf{j}$.

5) Let $T(x, y) = x^2y - xy^3 + 2$ and $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Compute $T_r(r, \theta)$, leaving your answer in terms of r and θ as usual.

6) Let $f(x, y, z) = xz\ln(xy^2\sin(z))$. Compute $f_z(x, y, z)$.

7) [20 pts] Answer T or F; you do not have to justify. Assume $f : R^2 \to R$.

If P has spherical coordinates $(4, \pi/2, \pi/2)$, it has rectangular coordinates $(0, 4, 0)$.

If ∇f exists at a point, then $f(x, y)$ is continuous there.

A bounded open set D in R^2 may contain one of its boundary points.

A nonzero gradient vector $\nabla f(P)$ is orthogonal to the level curve of f through P.

If D is a bounded set in the first quadrant of R^2, then D is open.

The tangent plane to $f(x, y) = x^2 + y$ at $(1, 1)$ has normal vector $\mathbf{n} = \langle 2, 1, -1\rangle$.

If the level curves of f are circles (or points) then the graph is a quadric surface.

There is a parametrized curve C with $\mathbf{T} = \mathbf{i}$, $\mathbf{N} = \mathbf{j}$ and $\mathbf{B} = \mathbf{k}$ when $t = 1$.

It is possible that $D_i f - D_j f > ||\nabla f||$ at some point.

It is possible that $|D_i f| + |D_j f| < ||\nabla f||$ at some point.

8) [10 pts] Consider $\lim_{(x,y)\to(1,2)} \frac{x+y}{x^2+4}$.

8a) Compute this limit along the line where $y = 2x$.

8b) Compute this limit along the line where $x = 1$.

8c) Do you think the original limit exists, as $(x, y) \to (1, 2)$ (not restricted to any particular line)? Justify your answer. You can include additional calculations, definitions or theorems.

Bonus [about 5 points]: Prove Thm 12.6.3c, that $\kappa = ||\mathbf{v} \times \mathbf{a}||/||\mathbf{v}||^3$.

Remarks and Answers: The average among the top 23 was approx 76%, which is good. The two highest scores were 95 and 90. The scores were especially good on problems 4 and 5 (approx 88%) but rather low on problem 2 (55%). An advisory scale for Exam II is

A's 84 to 100
B's 74 to 83
C's 64 to 73
D's 54 to 63
F's 00 to 53

To estimate your semester grade so far, average your two exam scores and apply the scale below. Your HW grades matter too, but are hard to factor in until later (see me if you need a rough estimate).

A's 80 to 100
B's 70 to 79
C's 60 to 69
D's 50 to 59
F's 00 to 49

1) $\mathbf{T}(t) = (1/5)\langle 3\cos(t), -3\sin(t), 4\rangle$, $\mathbf{N}(t) = \langle -\sin(t), -\cos(t), 0\rangle$ and $\mathbf{B}(t) = \langle 4\cos(t), -4\sin(t), -3\rangle$. The answer for the osculating plane is a little awkward because we are not given a specific value of t. But we can still use $n = \mathbf{B}(t)$ and $\mathbf{P} = \mathbf{r}(t)$ and get

$$4\cos(t)(x - 3\sin(t)) - 4\sin(t)(y - 3\cos(t)) - 3(z - 4t) = 0$$

2a) Graph $y = 2/x$ and $y = 3/x$ (parts of two hyperbolas) in the first quadrant of the $xy-$ plane, and label them.

2b) The standard example, done in class and the textbook, is $f(x, y) = \frac{xy}{x^2+y^2}$. If you came close enough to this, I gave partial credit. Ideally, you should define $f(0, 0)$ to be 0 (otherwise the partials do not exist), but I did not deduct points for missing that rather fine point.

Any other example was probably marked wrong without much comment. Feel free to bring those to me for discussion / explanation.

2c) $\nabla f = \langle 4, 5\rangle$. No calculation required. A common mistake was to answer with a scalar such as 4+5=9.

I graded 2a, 2b and 2c separately, and then wrote the sum of the three grades in the left margin.

3) $a_T = a_N = 2\sqrt{5}$. You do not need the formula $(a_T)^2 + (a_N)^2 = ||\mathbf{a}||^2$ but you could use it to check your work or maybe to take a shortcut.

4) $\mathbf{y}(t) = 3t^2/2\ \mathbf{i} - 2t^3/3\ \mathbf{j} + \mathbf{i} + \mathbf{j}$ which can be simplified a little.

5) $T_r = 3r^2 \cos^2\theta \sin\theta - 4r^3 \cos\theta \sin^2\theta$. The preferred method (because it works in many settings) is the chain rule, $T_r = T_x x_r + T_y y_r$. Since the wording here did not specify the method, it is also OK to eliminate x and y, starting with $T = (r\cos\theta)^2 \cdots$ etc.

6) Using the product rule and the chain rule (probably), get $f_x = x\ln(xy^2 \sin z) + xz \cot z$. Some people left the second term in the form $\frac{x^2y^2z\cos z}{xy^2\sin z}$, but anything this easy to simplify should be simplified. It is OK to use the $\ln(ab)$ formula here as a minor shortcut.

7) TFFTF TFTTF

8c) Since f is continuous at this point, the limit exists and is equal to $f(1,2) = 3/5$. So, the answer to both 8a and 8b is 3/5 (but with some work included). A fairly common mistake in 8c was to compute a limit along some line such as $y = x$ that does not even go through $P(1,2)$. The result is almost certainly not 3/5, is fairly meaningless, and is likely to mislead you.

B) See the text.