MAC 2313 Oct 24, 2019
Exam II Prof. S. Hudson

1) [5 points each] Let $f(x, y) = \ln(y - 2x)$. Answer these in the spaces provided.

a) Find the domain of f. Answer with words and/or formulas; a sketch is optional.

b) Find the range of f.

c) Find the boundary of the domain of f

d) Is the domain open or closed or neither ?

2) Find all the second order partial derivatives of $f(x, y) = x^2y + \sin y$.

3) Find the direction $\mathbf{u}$ in which $f(x, y, z) = \frac{x}{y} - yz$ increases most rapidly at $P(4, 1, 1)$. Then find $D_{\mathbf{u}}f(P)$.

4) Find the absolute extrema of $f(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangle R where $0 \leq x \leq 5$ and $-3 \leq y \leq 0$. Hint (to save you some time): they occur either on the interior or on the bottom edge of R, where $y = -3$. Find the locations and values of each.

5) Assuming that $x^2+xy+y^2-7 = 0$ defines y as a function of x, use implicit differentiation to find dy/dx at (1,2).

6) Evaluate the iterated integral $\int_0^4 \int_{-1}^1 (x - y)\ dy\ dx$.

7) [20 points] Answer T or F; you do not have to explain (but it might help, if there is some minor misunderstanding). Assume $f : R^2 \to R$ is differentiable.

The function $g(x, y) = \frac{x+y}{2+\cos x}$ is continuous on R^2.

The absolute minimum of $f(x, y) = 16 - 4x^2 - 4y^2$ on the open disk $x^2 + y^2 < 4$ occurs at infinitely many different points.

If f is differentiable on the xy-plane, then f is integrable on every rectangle in the plane.

If D is contained in a triangle in R^2 then D is bounded.

If D is contained in a closed triangle in R^2 then D is closed.

A nonzero gradient vector $\nabla f(P)$ is orthogonal to the level curve of f through P.

The tangential scalar component of acceleration is $a_T = \kappa||\mathbf{v}||^2$.

The linearization of the surface $z+2x+3y^3 = 8$ at $P(1, 2, 1)$ is $(x-1)+2(y-2)+(z-1) = 0$.

Lagrange's method (Ch.14.8) is based on the formula $\nabla \mathbf{f} = \lambda \nabla \mathbf{g}$.

The maximum value of $D_{\mathbf{u}}f(P))$ is $||\nabla f(P)||$.

8) Choose ONE proof and include enough words. If you use the back of the page, leave a note here.

a) Write out and prove formula (5) from Ch.13.5, which expresses κ in terms of velocity and acceleration.

b) By considering different paths show that the limit of $f(x,y) = \frac{x^4-y^2}{x^4+y^2}$ does not as exist as $(x,y) \to (0,0)$. I would not suggest the polar coordinate method (though it may be possible).

Bonus [about 5 points]: Show that if f is differentiable and $z = xf(x/y)$, then all tangent planes to the graph pass through the origin.

Remarks and Scales: The average among the top 20 was 63% with high scores of 87 and 85. The average scores were under 45% on problems 1 and 4. They were above 80% on problems 2, 6 and 8. The advisory scale for Exam II is

A's 72 to 100
B's 62 to 71
C's 52 to 61
D's 42 to 51

To help you estimate your semester grade, here is an advisory scale based on a weighted average of the 2 quizzes [5 points total], Exam I [15 points] and Exam II [20 points]. Using this scale, I wrote your average and resulting current grade in the upper right on your Exam II paper. Check that. I have not yet included your HW average since that is difficult to scale until the end, but you might make some rough adjustment (maybe ± 2 points) for that missing factor. Note that all this is an estimate; the precise definite scale is set at the end.

A's 67 to 100
B's 55 to 66
C's 46 to 54
D's 37 to 45

There were two versions of this exam, which differed slightly in problems 1, 2, 6 and 7. Most exams were Version A, the version above. If you had Version B, I put a star next to the 'Exam II' label near the top of your test before grading it. I have inserted remarks about Version B in the answers below as needed.

Answers:

1) Everybody knows that the domain of $y = \ln(x)$ is the set $x > 0$ and the range is the set of all real numbers.

1a) Domain f = $\{(x,y) | \; y - 2x > 0\}$. You might express this other ways, but your answer should be a subset of the xy-plane.

1b) Range f = $(-\infty, \infty)$, or you can say 'all real numbers'.

1c) Boundary = $\{(x, y)|\ y = 2x\}$. A picture is helpful here, but in many examples like this one, you can just change the $>$ to $=$ (practice with this and some caution are advised).

1d) Open. The domain does not include any boundary points, because if $y = 2x$ is true then $y - 2x > 0$ is false.

On version B of the exam, $f(x, y) = \ln(y - x)$. The answers are the same as above with no 2's. See also exercises 14.1.25 etc.

2) Start with $f_x = 2xy$ and $f_y = x^2 + \cos y$. Then the four answers, in any order, are $f_{xx} = 2y$, $f_{xy} = 2x$, $f_{yy} = -\sin y$, $f_{yx} = 2x$. Don't forget to include both mixed partials - though if you omitted one of those, the penalty was tiny. See 14.3.41 to 53.

On version B of the exam, $f(x, y) = x^2y + \cos y$ and then $f_{yy} = -\cos y$.

3) $\nabla f = \langle 1/y, -x/y^2 - z, -y\rangle = \langle 1, -5, -1\rangle$ at P. So, $\mathbf{u} = \frac{\nabla f}{||\nabla f||} = 27^{-1/2}\langle 1, -5, -1\rangle$. And $D_{\mathbf{u}}f(P) = ||\nabla f|| = 27^{1/2} = 3\sqrt{3}$.

I did not require simplifications such as $27^{1/2} = 3\sqrt{3}$, though that may be a good habit. It is OK but not necessary to use the formula $D_{\mathbf{u}}f(P) = \nabla f \cdot \mathbf{u}$. This formula was used to plan the given method, but it is not needed again. See 14.5.21.

4) Set $f_x = 2x + y - 6 = 0$ and $f_y = x + 2y = 0$ (so that $x = -2y$) and get the stationary point $x = 4$, $y = -2$. Get $z = f(4, -2) = -10$. The method is similar to Calc I, so far.

Set $y = -3$ (see the hint) so that $f = x^2 - 9x + 11$ on this line segment. And $0 \le x \le 5$. The next phase is a standard Calc I problem. Set $f' = \frac{d}{dx}(x^2 - 9x + 11) = 0$ and get $x = 9/2$ (you can use $f_x = 0$ for this step if you prefer). Then $z = f(9/2, -3) = 11 - 81/4 = -9.25$ (I used $x^2 - 9x + 11$, but the original $f(x, y)$ formula is OK). Including the endpoints as usual, $z = f(0, -3) = 11$ and $z = f(5, -3) = -9$.

The absolute max must exist, by the EVT. Trusting the hint, that no more work on the boundary is needed, the max must be one the four z's found above. So, it is $z = 11$, at $(0, -3)$. Likewise, the min is -10 at $(4, -2)$. This is exercise 14.7.35, with a hint.

5) You can solve this using Calc I implicit differentiation, applying $\frac{d}{dx}$ to both sides of the given equation, leading to $\frac{dy}{dx} = -4/5$. It is a little faster to use 14.4 Thm 8 with $\frac{dy}{dx} = -f_x/f_y = -4/5$. This is exercise 14.4.27 with slightly different wording.

6) 16. $\int_{-1}^{1} x - y\ dy = xy - y^2/2\ |_{-1}^{1} = 2x$, and then $\int_0^4 2x\ dx = 16$.

On version B of the exam, the last step is $\int_0^2 2x\ dx = 4$. This is exercise 15.1.2.

With practice, you may take some shortcuts on integrals like this. For example, $\int_{-1}^{1} y\ dy = 0$ because y is an odd function. This means you can delete the '$-y$' from the start. Also, $\int_{-1}^{1} x\ dy = 2x$ because x is a constant (and $1 - (-1) = 2$). So, you can get to $\int_0^4 2x\ dx$ with almost no work. But for grading purposes, always show some work or make a note of your reasoning.

7) TFTTF TFFTT. On version B of the exam, the same statements appear, but in a different order. The answers on B are TFTFT FTFTT.

8) See the textbook or lectures for 8a. For 8b (see 14.2.43), you can use an approach with $x = 0$ to get a path-limit, $L_1 = -1$. An approach with $y = 0$ gives a path-limit, $L_2 = 1$. Since they are different, the main limit doesn't exist.

Bonus) Let $P(x_0, y_0)$ be an arbitrary point and recall that $z = xf(x/y)$. We must handle this unusual use of 'f'. The tangent plane at P has the adjusted formula

$$z = x_0 f(P) + z_x dx + z_y dy = x_0 f(P) + [f(P) + (x_0/y_0) f'(P)](x - x_0) + (-x_0^2/y_0^2) f'(P)(y - y_0)$$

using the product rule and chain rule. To check the claim, we set $x = 0$ and $y = 0$. After some canceling, we get $z = 0$, which shows the tangent plane contains $(0, 0, 0)$. Done.

My bonus questions tend to be more challenging than the others, but they are worth a try if you have extra time on the exam.