MAC 2313 Mar 10, 2021
Exam II Key Prof. S. Hudson

1) Let $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 3t\mathbf{k}$. Write $\mathbf{a}$ in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ without finding $\mathbf{T}$ and $\mathbf{N}$.

2) [5 points each] Let $f(x,y) = \ln(y-x)$. a) Find the domain of f. Express your answer as a set.

2b) List two elements of the range of f.

2c) Let D be the rectangle where $0 < x \le 2$ and $1 \le y < 3$. Give an example of a boundary point of D that is not in D.

2d) Give an example of an interior point of D.

3) Find the absolute extrema of $f(x,y) = x^2 + 2y^2 - 2x - 8y + 5$ on the square R where $0 \le x \le 3$ and $0 \le y \le 3$. Hint (to save you some time): they occur either on the interior or on the edge where $x = 3$. Find the locations and values of each.

4) Find all the second order partial derivatives of $f(x,y) = xy^2 + \sin(xy)$.

5) Find the equation of the tangent plane to the surface $3z - x^2 = 0$ at the point (2,0,2).

6a) Consider $f(x,y,z) = \sin(xy) - yz + e^x z$. Find the direction in which the function increases most rapidly at the point $(1,0,1)$. Answer with a unit vector.

6b) (same f as in 6a) If $x = e^u v$, $y = v\sin(u+v)$, and $z = v^2$, find f_u.

7) [20 pts] Answer True or False. Assume $f : R^2 \to R$ is differentiable in this problem and in the next (the proof).

The function $g(x,y) = \frac{x+y}{1-2\cos xy}$ is continuous on R^2.

The absolute minimum of $f(x,y) = (x-y)^2$ on the open disk $x^2+y^2 < 4$ occurs at infinitely many different points.

If f is differentiable on the xy-plane, then f is integrable on every rectangle in the plane.

The linearization of $z + 2x + 3y^3 = 8$ at $P(1,2,1)$ is $2(x-1) + 36(y-2) + (z-1) = 0$.

If f_x and f_y exist at P, then $f(x,y)$ is differentiable at P.

If D is an open set in the first quadrant of R^2, then D is bounded.

If f_{xy} and f_{yx} are continuous on R^2 they are equal.

It is possible that $D_u f > ||\nabla f||$ at some point.

At every point, $D_i f + D_j f = \nabla f \cdot \langle 1, 1 \rangle$.

$\lim_{(x,y)\to(0,0)} \frac{x}{x} = 1$.

8) [10 pts] Choose ONE and prove it. This means formulas and *words*, probably no examples.

a) The gradient of f at P is normal to the level curve of f at P.

b) $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$.

Bonus) [Hard] Give an example of a function $f(x,y)$ such that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, but the limit is equal to 0 along any straight line path to (0,0).

Remarks, Answers: The average among the top 12 was 63, which is down about 4 points from Exam I, but not really bad. The highest grades were 82 and 81. The grades on problem 5 and 6 were very good (80%), but they were low on problem 1 (40%). Here is an advisory scale for Exam 2:

A's 71 - 100
B's 61 - 70
C's 51 - 60
D's 41 - 50

You can use the same scale for your semester grade so far, based on 3 quiz grades at 2 points each, and two exams at 15 and 20 points, resp. For example suppose your quiz grades are 40, 50 and 60, and your Exam 1 and 2 grades are 60 and 80. You would compute 2(40+50+60) + 15(60) + 20(80) and divide that by 41 to get your average, 68, which is a C or C+. I will write that average on your Exam 2 paper if I have time. You might make some adjustment to that, based on your HW grades, etc, but I find that hard to do systematically at this point.

1) We need to find a_T and a_N from standard formulas. $a_T = \frac{d}{dt}\frac{ds}{dt} = \frac{d}{dt}||\mathbf{r}'|| = 0$ (check that $||\mathbf{r}'||$ is a constant). Also $a_T^2 + a_N^2 = ||\mathbf{a}||^2$ (this is based on a picture and the Pythagoras theorem). So $a_N = ||\mathbf{a}|| = 2$. $\mathbf{a} = 0\mathbf{T} + 2\mathbf{N}$. See 13.5.1.

2a) $D = \{(x,y) : y - x > 0\}$. It is also OK to express this in words.

2b) Any two real numbers are OK, such as ln(1/2) and 17, for example (I intended to make this one a little harder, but forgot to do so).

2c) $P(0,3)$ is one of the corners which is not in D, but there are many other good answers.

2d) $Q(1,2)$ is the center, but again there are many other answers.

3) To find stat pts, set $0 = f_x = 2x - 2$ and $0 = f_y = 4y - 8$ and find only the point $(1,2)$ (where $f = -4$). To find boundary pts, set $x = 3$ (from the hint to ignore the other edges) so that $f = 8 + 2y^2 - 8y$ and $0 \le y \le 3$. We solve this Calc I max-min problem by setting $0 = \frac{d}{dy}8 + 2y^2 - 8y = 4y - 8$ getting $y = 2$ (very similar to the previous step, but that doesn't always happen). This boundary pt is $(3,2)$ where $f = 0$. Also include the endpoints of $0 \le y \le 3$ (as in most Calc I examples) to get the points (3,0) where $f = 8$ and (3,3) where $f = 2$.

Answers: The maximum value is 8 at (3,0). The minimum is is -4 at (1,2). If you noticed that this function is the distance to (1,2) [or a variation of that, actually] you might be able to guess these two points from a picture of D. But you should still give some work and reasoning.

4) Start with
$f_x = y^2 + y\cos(xy)$ (chain rule) and
$f_y = 2xy + x\cos(xy)$. Then the answers,
$f_{xx} = -y^2\sin(xy)$,
$f_{xy} = 2y + \cos(xy) - xy\sin(xy)$ (product rule),
$f_{yy} = 2x - x^2\sin(xy)$ and
$f_{yx} = 2y + \cos(xy) - xy\sin(xy)$ (which equals f_{xy}).

5) The gradient of $f(x, y, z)$ is $\langle -2x, 0, 3\rangle = \langle -4, 0, 3\rangle$, so the equation is $-4(x-2)+3(z-2) = 0$. Simplification is optional for these.

6a) The gradient of $f(x, y, z)$ is $\langle y\cos(xy) + e^x z, x\cos(xy) - z, -y + e^x\rangle = \langle e, 0, e\rangle$. Normalizing this, $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle$.

6b) $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} = [y\cos(xy) + e^x z]e^u v + [x\cos(xy) - z]v\cos(u+v) + 0 = [v\sin(u+v)\cos(e^u v^2\sin(u+v)) + e^{e^u v}v^2]e^u v + [e^u v\cos(e^u v^2\sin(u+v)) - v^2]v\cos(u+v)$.

7) FTTTF FTFTF

8) See the text or lectures. Most people chose b and did ok if they included enough words.

Bonus) One student got this, with an answer like $\frac{x^3-y^5}{x^2-y}$. They did not explain it but I think it is correct and gave 5 points. The limit does not exist along the parabola where $y = x^2$.

There were several incorrect guesses like $\frac{x-y}{x-1}$ which is continuous at (0,0). And some others like $\frac{x-y}{x}$ which does not have a limit along the path $x = 0$. The last TF is false for a similar reason.