MAC 2313 Nov 8, 2018
Exam III Prof. S. Hudson

1) Short answer (5 pts each)

1a) Find a unit normal vector to the level curve of $f(x,y) = xy^2$ at $P(1,2)$.

1b) Find the Jacobian J for $x = u + 4v$, $y = 3u - 5v$. (alternative notation - find $J = \partial(x,y)/\partial(u,v)$ for the transformation $T(u,v) = \langle u + 4v, 3u - 5v\rangle$).

1c) Let R be the region bounded by $y = x^2$ and $x = y^2$. Express $\int\int_R f(x,y)\ dA$ as an iterated integral of the form $\int\int f(x,y)\ dx\ dy$ (but include the limits).

1d) Evaluate $\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^x xy\ dy\ dx\ dz =$

2) The parametric equations below define a cone that results from revolving the line $y = x$ in the xy-plane around the x-axis. Use these and a double integral to find the surface area of the portion of the cone for which $0 \le u \le 2$ and $0 \le v \le 2\pi$.

$x = u$, $y = u\cos v$, $z = u \sin v$

3) Find the absolute extrema of $f(x,y) = xy - 2x$ on the triangular region R, with vertices at $(0,0)$, $(0,4)$ and $(4,0)$. State what the max and min values are (if they exist) and where they occur.

4) Evaluate this integral by reversing the order of integration: $\int_0^2 \int_{y/2}^1 e^{x^2} dx\ dy$. Include a picture of the region R.

5) [20pts] True-False. Let S be the region where $1 < x < 2$ and $0 < y < g(x)$ where $g(x)$ is some unknown positive continuous function. (So, for example, the area of S is $\int_1^2 g(x)\ dx$). Assume all functions mentioned below are differentiable.

The function $f(x,y) = yx^3 + x + y^2 + 3$ has no relative extrema on S.

The centroid of S must lie in S.

If a level curve of $f(x,y)$ and a level curve of $h(x,y)$ both pass through the point $(3,1)$ then $\nabla f(3,1)$ is a scalar multiple of $\nabla h(3,1)$.

The centroid of the region bounded by $r = 1 + \cos\theta$ is the origin.

The average value of $f(x,y,z) = xyz$ over the spherical region $\rho \le 5$ is zero.

$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} z\ dz\ dy\ dx = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho \cos\phi\ d\rho\ d\theta\ d\phi$

$\int_0^{\pi} \int_0^{\pi/2} \int_0^1 \rho\ d\rho\ d\theta\ d\phi = 2\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho\ d\rho\ d\theta\ d\phi$

$\int_0^1 \int_{-2}^2 \int_0^3 x^2 + y^3 + z^4\ dx\ dy\ dz = 2\int_0^1 \int_0^2 \int_0^3 x^2 + y^3 + z^4\ dx\ dy\ dz$.

If $\nabla f = \langle 3,\ 4\rangle$ at P and $\mathbf{u}$ is any unit vector, then $D_{\mathbf{u}} f(P) \le 4$.

Lagrange multipliers are mainly used to solve optimization problems.

6) The function $f(x,y) = 4xy - x^4 - y^4$ has 3 critical points. One is $(-1,-1)$. Find the other two, and use the Second Partials Test to classify each of the three (as a rel.max, a

rel.min or a saddle point).

7) Use spherical coordinates to find the volume of the solid bounded above by the sphere $\rho = 4$ and below by the cone $\phi = \pi/3$. You can leave your answer as a triple integral - you do not have to evaluate it.

8) Choose ONE, circle it and do it.

a) Prove Thm 13.7.2 as done in the text: if f is differentiable at (x_0, y_0) then the tangent plane to the graph $z = f$ at (x_0, y_0) is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

b) State and prove the main theorem about Lagrange multipliers, Thm.13.9.3. Your answer does not have to be quite as precise as usual, but explain the formula in detail, probably including pictures, as done in the text.

Bonus) [about 5 points] Let R be the region in the xy-plane where $y \geq 0$ and $1 \leq r \leq 2$. Let S be the rectangular region in the uv-plane where $1 \leq u \leq 2$ and $0 \leq v \leq \pi$. Find a transformation $T : R \to S$. Give a formula for T.

Remarks and Answers: The average among the top 23 was 70. The highest scores were 92 and 86. The results were similar for most problems, but very high on #4 (95%) and a bit low on the TF (58%). Here is an advisory scale for the Exam:

A's 78 to 100
B's 68 to 77
C's 58 to 67
D's 48 to 57

Here is an advisory scale for the term so far - for the exam average, not yet including the HW;

A's 79 to 100
B's 69 to 78
C's 59 to 68
D's 49 to 58

1a) $\mathbf{n} = \sqrt{1/2}\langle 1, 1\rangle$.

1b) 17.

1c) $\int_0^1 \int_{y^2}^{\sqrt{y}} f(x, y)\ dx\ dy$. Several people had the limits y^2 and $\sqrt{y}$ reversed. A labeled picture should help you get it right. Also notice that $y^2 < \sqrt{y}$ when $y = 1/4$, for example.

1d) 81/5.

2) Area $= \int_0^{2\pi} \int_0^2 ||\mathbf{r}_u \times \mathbf{r}_v||\ du\ dv = \int_0^{2\pi} \int_0^2 \sqrt{2}\ u\ du\ dv = 4\sqrt{2}\pi$. Several people mistakenly converted the cross product into a scalar and/or forgot to take the norm before integrating.

3) Max $= f(1,3) = 1$ and Min $= f(4,0) = -8$. The standard steps are to draw a picture and label it, find the stationary points (there are none, unless you count (0,2), but that point is on the boundary, and is not especially important) and then the max and min on each line segment of the boundary (and the segment $y = 4 - x$, with $0 \le x \le 4$, leads to the two answers). The most common problem was not handling the boundary segments properly.

A picture was not required but people who included one seemed more likely to notice the 3 segments and to deal with them. This was a Ch. 13.8 problem and does not require the Lagrange method, but a few people used that successfully.

4) $e - 1$. See the text.

5) TFFFT FTFFT

6) Solve $\nabla f(x,y) = \mathbf{0}$ to get the 3 critical points. There are rel max's at $(1,1)$ and $(-1,-1)$ because $D > 0$ and $f_{xx} < 0$ at both points. There is a saddle point at $(0,0)$ because $D < 0$ there.

7) $V = \int_0^{\pi/3} \int_0^{2\pi} \int_0^4 \rho^2 \sin\phi \; d\rho d\theta d\phi$.

8) People preferred 8a by a 2-1 margin. See the text or lecture notes for the proofs. Be sure to include *words* to explain the ideas - not necessarily the same words as the book, but approximately as many.

8a) Key ideas: Introduce $F(x,y,z) = z - f(x,y)$ (or use $f - z$), so that $F = 0$ on the graph of f that we are studying. Do not use the x_0 notation just yet. Mention that this is a level surface of F [so that Def 13.7.1 applies], so that ∇F is a normal vector for the tangent plane.

The main problem on 8a was inadequate explanation; omitting phrases such as *level surface* , *normal vector* and ∇F. The author avoids some of this by quoting equation (3) twice, which is Def 13.7.1, but if you can't do that, you need to put the idea into words.

8b) It is OK to explain this in R^2, which is easier, partly for pictures. Introduce f and g and explain why the level curves at P are tangent to each other. This part does not have to be very precise, but you should at least mention the level curves and the tangency. Then a theorem implies the gradients are parallel, hence the λ formula.

Bo) This is 14.7.31 but without the picture, and with u, v reversed (probably by accident). So that you can follow the book version more easily I am answering that version here (so $1 \le v \le 2$ instead of u, etc). You should draw the picture yourself. It suggests polar coordinates, so we can set $v = r$ and $u = \theta$, which matches well. This more or less defines T, except that we are asked for an explicit formula. We know $x = r\cos\theta = v\cos u$ and $y = v \sin u$. For most 14.7 problems these might be enough, since usually T goes from u,v to x,y. But the problem asks for it the other way, so we use the old conversion formulas from polar to rectangular,

$$T(x,y) = \langle \tan^{-1}(y/x), \sqrt{x^2+y^2} \rangle$$

This answer still has a problem, since y/x is undefined when $x = 0$, and that does happen in our region. We could just say that $u = \pi/2$ in that case. Or, see the textbook answer key for a slightly-messy alternative formula for u.

For the version on the exam, use $T(x, y) = \langle \sqrt{x^2 + y^2}, \tan^{-1}(y/x) \rangle$, but fix it up a bit, as mentioned above.