MAC 2313 Nov 21, 2019
Exam III Prof. S. Hudson

The first 5 problems are 7 points each. Problems 6-9 and 11 are 10 points each.

1) Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.

2) Change to an equivalent polar integral and evaluate, $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2) dx\ dy$.

3) Evaluate $\int_0^\pi \int_0^\pi \int_0^\pi \cos(u + v + w)\ du\ dv\ dw$.

4) Sketch the graph of $\phi = \pi/6$ (spherical coordinates) in R^3.

5) Evaluate $\int_C (x + y)\ ds$ where C is the straight line segment $x = t$, $y = 1 - t$, $z = 0$ from $(0, 1, 0)$ to $(1, 0, 0)$.

6) Evaluate $\int\int_R (y - 2x^2)\ dA$ where R is the square region bounded by $|x| + |y| = 1$. To save time, you may use the symmetry of y and of x^2 (two different symmetries).

7) Find the center of mass of a solid of constant density δ bounded below by the disk R, $x^2 + y^2 \le 4$ in the plane $z = 0$, and above by the paraboloid $z = 4 - x^2 - y^2$. As usual, you are allowed to use symmetry and other coordinate systems.

8a) Solve the system $u = x - y$, $v = 2x + y$ for x, y. Then, find the Jacobian $\partial(x, y)/\partial(u, v)$.

8b) Use 8a to evaluate $\int\int_R (2x^2 - xy - y^2)\ dx\ dy$, for the region R in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$ and $y = x + 1$.

9) Find the work done by $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$ over the curve C in the direction of increasing t, given that $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$.

10) [15 points] Answer T or F; you do not have to explain (but it might help, if there is some minor misunderstanding).

- If f is continuous on R^2, then $\int_0^3 \int_0^3 f(x, y) dydx = \int_0^3 \int_0^3 f(y, x) dxdy$.
- If D is a solid contained in a sphere S in R^3, then the centroid of D lies in S.
- The Jacobian of a transformation is always positive.
- If R is any rectangle or disk in R^2 then $\int\int_R (4 - x^2 - y^2)\ dA \le 8\pi$.
- If R is a disk in R^2 then the average value of $4 - x^2 - y^2$ over R is at most 2.

11) Circle ONE to do. If you use the back of the page, leave a note here.

a) State and prove equation 3 from Ch.14.6 about the equation of the tangent plane to a graph $z = f(x, y)$.

b) Explain (justify) the formula $r\ dr\ d\theta$ used in polar coordinate integrals.

Bonus [5 points]: Compute the Jacobian J for the transformation to spherical coordinates ($\rho^2 \sin\phi$) using the determinant method.

Remarks and scales: Most of the problems were from the textbook exercises, or slight variations of them. See 15.3.21, 15.4.11, 15.5.17, 15.7.14, 16.1.9, 15.2.55, 15.6 Ex.1, 15.8.1, 15.8.6, and 16.2.19. The average was 56 with high scores of 84 and 78. The worst scores by far were on problem 6 (20%) and the best were on 2 (80%) and 11 (88%). The advisory scale is:

A's 67 to 100
B's 54 to 66
C's 43 to 53
D's 33 to 42

I have not had time to recalculate your semester averages, but I don't think this exam will change the scale from the one on the Exam 2 key.

Answers:

1) $\int_0^2 \int_0^2 x^2 + y^2 \; dx \; dy = \cdots = 32/3$ and then divide by 4 (the area). So, $f_{avg} = 8/3$.

2) $\int_0^{\pi/2} \int_0^2 r^2 \; r \; dr \; d\theta = \cdots = 2\pi$. The most common basic mistake was failure to draw a picture. This was not required for full credit, but it's strongly recommended for these problems.

3) 0, various ways. The inner integral equals $\sin(u+v+w)|_0^\pi$. You can simplify later using the identity $\sin(\pi+\theta) = -\sin\theta$, if desired. You cannot use $\cos(u+v+w) = \cos(u)+\cos(v)+\cos(w)$ which not true in general. Test it with $u = v = w = \pi/3$, for example.

4) It looks like an ice cream cone (but extending upwards forever). As usual with one equation in R^3, it is a 2D surface.

5) $\int_0^1 1 \; \sqrt{2} \; dt = \sqrt{2}$. The main steps are $x+y = t+(1-t) = 1$, $ds = \frac{ds}{dt}dt$, $\frac{ds}{dt} = ||r'|| = \sqrt{2}$ and finding t at the endpoints of C. It is also OK to take a shortcut based on $\int_C ds =$ length of $C = \sqrt{2}$.

6) -2/3. Draw a picture to help find symmetries and the limits of integration. Most people did not do this. (see the next paragraph for more on this). Since y is an odd function, $\int\int_R y \; dA = 0$. Since x^2 is even (wrt both x and y) $\int\int_R x^2 \; dA = 4\int\int_T x^2 \; dA$ where T is the part in quadrant 1, the triangle under $x+y=1$. So, the answer is
$(-2)4\int_0^1 \int_0^{1-x} x^2 \; dy \; dx = \cdots = -2/3$.

The picture. R is a tilted square like this $\diamondsuit$, with corners at (0,1), (1,0), (0,-1), (-1,0). These points should be easy to find by trial and error. You can also plot more, such as (1/2, 1/2) if you like. An alternative method is to graph just the part where the variables are positive (the line segment x+y=1 in quadrant 1) and then mirror images of that in the other 3 quadrants (because the absolute value function is even). In the drawing process, you should notice the symmetries wrt x and y which I used above to simplify the set-up.

7) $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$. As usual, it is helpful to draw a picture, and we notice that this is a solid of revolution. We see that the center of mass must be on the z axis, so $\bar{x} = \bar{y} = 0$. The integrals will be done with cylindrical coordinates. Set $\bar{z} = M_{xy}/M$.

$M_{xy} = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} z\ r\ dr\ d\theta = \cdots = 32\pi/3$ and similarly $M = \cdots = 8\pi$, so $\bar{z} = 4/3$.
It seemed that many people did not study this topic and did not include basics like $\bar{x}$, M, etc.

8a) $x = (u+v)/3,\ y = (-2u+v)/3,\ J = 1/3.$

8b) $\int_{-1}^2 \int_4^7 uv\ J\ dv\ du = \cdots = 33/4$. Factoring the integrand into uv may be a little tricky the first time, but we did this example in class.

9) $W = \int_C \mathbf{F} \cdot \mathbf{dr} = \int_0^1 2t^3\ dt = -1/2.$

10) TTFTF

11) See the text. For 11b I accepted either the picture-based proof or the determinant method (which requires less explanation).

Bonus) Nobody got this to the end. It is a 3x3 determinant, with a lot of routine calculation and simplification.